UNIFIED FUNCTIONAL METHOD FOR SOLVING GENERAL POLYNOMIAL EQUATIONS OF DEGREE LESS THAN FIVE

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A unified method for solving general polynomial equation of degree less than five that incorporate a computational formula that relate the coefficients of the depressed equation and the coefficient of the standard polynomial equation is proposed in this study. This is to ensure that this method is valid for all degree less than five. It shall apply the undetermined parameter method of auxiliary function to obtain solutions to these polynomial equations of degree less than five in one variable. In particular, the result of our work is a unification and improvement on the work of several authors in the sense that only applicable for the case of polynomial equation of degree one. Finally, our results improve and generalize the result by applying standard formula methods for solving higher degree polynomials. It is recommended that the effort should be made toward providing other variant methods that are simpler and friendly.

1.   Introduction

Considering how important the solution to polynomial equations is in the field of sciences (i.e., Mathematics, Statistics and Physics) and social sciences (i.e., Economics and Business Administration) some researchers have developed a unify method of solution to those polynomial equations that are solvable by radicals. Kulkarni (2006), Kalman & White (2001), provide an interesting, unified approach, based on undetermined parameter method, circulant matrices method respectively, for solving polynomial equations of degree four or less. However, it is important to note that, ironically, the methods developed by this researcher is not valid for solving general polynomial equations of degree one. Indeed, literature indicates that solution to polynomial equations have been investigated for centuries. Linear and quadratic polynomial equations were solved in the fifteenth centuries while the cubic and quartic polynomial equations were comprehensively solved in the sixteenth centuries.

Cubic equation was known since the ancient times, even by the ancient Greeks and the ancient Babylonians and the ancient Egyptians. In the 11th century, the famous Mathematician

Omar Khayyam discovered a geometrical method to solve cubic equation which could be used to get numerical answer by intersecting a parabola with a circle, and by using this method he found cubic equation can have more than one solution. He could not find algebraic formula for the general, but he could only solve cubic equation geometrically, Conner (1956).

As in Conner (1956), Scipione del Ferro discovered a formula that solved the so called "depressed cubic". Instead of publishing his solution, Del Ferro kept it a secret until his deathbed telling his student Antonio Fior. Niccolo Fotana also known as Tartaglia who solved many special cases of cubic equations and later reveals his techniques to Cardan. Girolamo Cardan was the one who gave a complete solution to the general cubic equation in his book, *The Great Art, or the Rules of Algebra* (Cardano, 1545). In that book *the Ars magma,* Cardano introduced the technique of substitution by Ludovico Ferrari that not only solved the cubic and quartic but became indispensable in polynomial algebra (Conner, 1956).

It may interest you to know that the first attempt to unify solutions to quadratic, cubic and quartic equations date at least to Lagrange (1869). Lagrange's analysis characterized the general solutions of the cubic and quartic cases in terms of permutations of the roots, laying a foundation for the independent demonstrations by Abel and Galois of the impossibility of solutions by radicals for general 5th degree or higher equations. Since then, researcher have pitched their tent in providing several other alternative (analytic) methods for the solution of general polynomial equation of degree less than five has been proposed in the literature. However, any work done in line with the purpose of proffering a unified functional method for solving $f_n(x) = 0$ $(n < 5)$ in radical were unaware.

Nickalls (2000) used differentiation to obtain $-b/na$ which he called the N-point of a polynomial, that is the point to which the axis must be moved to make the sum of the roots equal to zero. Futhermore, observe that what the N-point represents depends on the degree of the polynomial, in particular, N-point represents the root of a linear equation when $n = 1$, the turning point of a quadratic equation when $n = 2$, the point of inflection of a cubic equation when $n = 3$ etc. Das (2014) employed differential calculus method (N-point property) which depend on cardano's and Ferrari's method to obtain solution to cubic and quartic polynomial equation. Tiruneh (2019, 2020) introduced the functional method of solution to quadratic, cubic respectively via certain transformations and differentiation. However, it is important to note that the case of functional method of solution to quartic equation is yet to be announce, this will be taken care of as a particular case in our unified functional method for solving $f_n(x) = 0$ $(n < 5)$ in radical via certain auxiliary function.

Solving polynomial equation via reduction to depressed equation (that is polynomial equation without the second highest term) over the years has seemingly become a standard approach. Furthermore, the coefficient of this depressed equation plays a fundamental role in determining the solution of the standard polynomial equation. Now, the following depressed equations for each of the polynomials were recalled.

Quadratic: $y^2 + p = 0; \ p = -\left(\frac{b^2 - 4ac}{4a^2}\right)$

Cubic: $y^3 + py + q = 0; \ p = \frac{c}{a} - \frac{b^2}{3a^2}, q = \frac{d}{a} + \frac{2b^3}{27a^3} - \frac{bc}{3a^2}$

Quartic: $y^4 + py^2 + qy + r = 0; \ p = \frac{c}{a} - \frac{3b^2}{8a^2}, q = \frac{d}{a} - \frac{bc}{2a^2} + \frac{b^3}{8a^3}, r = \frac{e}{a} - \frac{bd}{4a^2} + \frac{b^2c}{16a^3} - \frac{3b^4}{256a^4}$

where $a, b. c, d$ are the coefficients (constants) of the associated polynomial equations and p, q, r are coefficients (constants) of the corresponding depressed polynomial equations.

Probably, it is of interest to note that except for the quadratic equation, the coefficient (constant) terms in these depressed equations are not readily handy to remember during computation since the solution to their associated general polynomial equation depend on them (this coefficients). Hence, it is our interest in this research work to develop a novel formula that completely determine the coefficients of these depressed equations for a general polynomial equation of degree n and then apply it to solve standard polynomial equation of degree less than five which were believed is yet to be announced in literature. Thus, the remaining part of this work is structured as follows: Section 2 is the methodology where it defines some concepts and proves the important theorems. Section 3 is the results of the polynomial, namely, linear, quadratic, cubic, and quartic and further explains the applications of the theorem. Finally, Section 4 concludes the paper by highlighting the future work that can be made to improve the research.

2. Methodology

It is important to note that Kulkarni (2006) introduced unified method for solving general polynomial equations of degree less than five by seeking for the zero solution of the polynomial (auxiliary function) of degree n $(n < 5)$ given by

$$g(x; b_0, \cdots b_{m-1}, c_0,, \cdots c_{m-1}, p) = \frac{(V_m(x))^k - p^k (W_m(x))^k}{1 - p^k}$$

where $V_m(x)$ and $V_m(x)$ are constituent polynomials of degree m, such that $m < n$ and p is unknown to be determined. The integer k has to satisfy the relation $km = n$ so that the auxiliary function will be of degree n. It is important to observe that in this method, when n is even, the number of unknowns is one more than the number of equations, this is an issue to worry about. However, in such case, Kulkarni remarked on the needs to assign some convenient value to an extra (one) unknown for determining the unknowns by solving the n equations.

This remain an issue since there is no lay down procedure for assigning this so-called convenient value to a particular unknown in the system of the equations. Furthermore, it is some worth ironical to observe that the auxiliary function constructed by Kulkarni is not applicable to $\sum_{j=0}^{n} a_{n-j} x^{n-j} = 0; a_n \neq 0; if\ n = 1$ which contradict the claim that it is valid for $n < 5$, since $n = 1 < 5$ and this is easily seen from the equation that determine the number of unknown which is given by

$$2m + 1 = \begin{cases} n & ;for\ n\ odd \\ n+1 & ;for\ n\ even \end{cases}$$

This has no solution for $n = 1$, since this implies that $m = 0$, suggesting that the auxiliary function is a constant. This cannot be, thus a contradiction. Furthermore, observe that for every $k > 1, km = n\ and\ n = 1$ implies that $1 = n = km = k(0) = 0 \Rightarrow\Leftarrow$ again, a contradiction. In order to do away with this anomaly, an alternative approach to the auxiliary function (polynomial) constructed by Kulkarni shall be introduced.

In this section, it is defined some important terms, definitions, lemmas, and theories as will be needed in the presentation of this research work. Let $\mathbb{N}$ be the set of natural numbers and C the set of real (or complex) numbers. Then for any given $n \in \mathbb{N}$, a function $f_n: C \rightarrow C$ is said to be a polynomial of degree n if there exists a constant $a_j (j = 0, 1, 2, \ldots, n)$ such that f is given by

$$f_n(x) = \sum_{k=0}^{n} a_k x^k; \quad a_n \neq 0 \tag{1}$$

Equation (1) is said to be depressed if $a_{n-1} = 0$, monic if $a_n = 1$ and the set of first n positive integers is defined by

$$[n] := \{1, 2, 3, \cdots, n\} \tag{2}$$

LEMMA 2.1

Let $f_n(x)$ be as in equation (1) and $\beta_n \in C$, if $x = \beta_n + y$ then

$$f_n(\beta_n + y) = \sum_{k=0}^{n} \left(\sum_{j=0}^{n-k} \binom{k+j}{j} a_{k+j} \beta_n^{\ j} \right) y^k \tag{3a}$$

Proof:

By hypothesis

$$f_n(x) = \sum_{k=0}^{n} a_k x^k$$

$$\Rightarrow f_n(\beta_n + y) = \sum_{k=0}^{n} a_k (\beta_n + y)^k = \sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k} \binom{k}{j} \beta_n^{\ k-j} y^j \right) =$$

$$a_0 \beta_n^{\ k} + a_1 \left(\sum_{j=0}^{1} \binom{1}{j} \beta_n^{\ 1-j} y^j \right) + a_2 \left(\sum_{j=0}^{2} \binom{2}{j} \beta_n^{\ 2-j} y^j \right) + \cdots + a_n \left(\sum_{j=0}^{n} \binom{n}{j} \beta_n^{\ n-j} y^j \right)$$

After some algebraic simplification, the required result is obtained. This completes the proof.

LEMMA 2.2

Let $f_n(x)$ be as in equation (1) and $\beta_n \in C$, then there exist a functional $f_n^{(k)} \left(\frac{d^k f_n}{d\beta^k} = f_n^{(k)} \right)$ such that

$$f_n^{(k)}(\beta_n) = k! \sum_{j=0}^{n-k} \binom{k+j}{j} a_{k+j} \beta_n^{\ j} \tag{3b}$$

Proof:

Since $f_n(\beta_n + y) = \sum_{k=0}^{n} p_{n,k} y^k$, where $p_{n,k} = \left(\sum_{j=0}^{n-k} \binom{k+j}{j} a_{k+j} \beta_n{}^j \right), p_{n,n} = a_n$, Observe that $f_n(\beta_n + y)$ is a polynomial of degree n and hence its Taylor's series expansion at the origin (Maclaurin's) is given by

$$f_n(\beta_n + y) = \sum_{k=0}^{n} \frac{f_n^{(k)}(\beta_n)}{k!} y^k \tag{3c}$$

Using lemma 2.1, the required result is obtained. This completes the proof.

LEMMA 2.3

Let $f_n(x)$ be as in equation (1) then there exists $\beta_n \in C$, such that the transformation $x = \beta_n + y$ implies that

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-2} x^{n-2} + a_{n-1} x^{n-1} + a_n x^n = 0$$

take the form

$$p_{n,0} + p_{n,1} y + p_{n,2} y^2 + \cdots + p_{n,n-2} y^{n-2} + a_n y^n = 0$$

Proof:

Recall; $f_n(x) = f_n(\beta_n + y) = 0$. From lemma 2.1 observe that

$f_n(\beta_n + y) = \sum_{k=0}^{n} p_{n,k} y^k$, where $p_{n,k} = \sum_{j=0}^{n-k} \binom{k+j}{j} a_{k+j} \beta_n{}^j$

Clearly, from definition $p_{n,n} = a_n$. Also the value of β_n that will guarantee that the coefficient of y^{n-1} vanishes implies that

$$\sum_{j=0}^{1} \binom{n-1+j}{j} a_{n-1+j} \beta_n{}^j = 0; \implies a_{n-1} + n a_n \beta_n = 0; \implies \beta_n = -\frac{a_{n-1}}{n a_n}$$

This completes the proof.

Remark 1: It is important to note that most standard proof for Lemma 2.3 readily apply the Newton's power sums formula Adamchik & Jeffrey (2003), which is rather complicated when compared with the method of our proof.

Now, combining equation (3b) and equation (3c), the depressed equation is obtained

$$f_n(\beta_n + y) = \sum_{k=0}^{n} f_n^{*(k)}(\beta_n) y^k \tag{3d}$$

where $f_n^{*(k)}(\beta_n) = \sum_{j=0}^{n-k} \binom{k+j}{j} a_{k+j} \beta_n{}^j$ is now considered as $p_{n,k}^* = \frac{1}{a_n} \sum_{j=0}^{n-k} \binom{k+j}{j} a_{k+j} \beta_n{}^j$.

Then, an equivalent depressed monic polynomial is obtained

$$g(y; u, v, w) = (vy + w)^m - (y^{r_n} + u)^m : \text{for some } r_n \in [n] \tag{4a}$$

$$x = y + \beta_n \tag{4b}$$

$$mr_n = n \tag{4c}$$

$$f_n(\beta + y) = \sum_{k=0}^{n} p^*_{n,k} y^k \tag{4d}$$

THEOREM 2.4

Let $n, m \in \mathbb{N}$ and $f_n(x) = 0$ a polynomial equation of degree n ($n \leq 4$), then there exist u, v, w an undetermined parameter such that the zero solution to the auxiliary function solves the given polynomial equation $f_n(x) = 0$

The above theorem for the case of linear, quadratic, cubic and quartic polynomial shall be proven in the Section 3.

3. Results and Applications

3.1 The Linear Case

It is clear that when $n = 1$, then $f_1(x) = f_1(\beta_1 + y) = 0$ implies that $\sum_{k=1}^{1} p^*_{1,k} y^k = 0$. It may simply write this as $p^*_{1,1} y = 0$.

Now, for $n = 1$ there is at least one $r_1 \in [1] = \{1\}$ such that $mr_1 = 1$. Since $\{1\}$ is a singleton set, it suffices to take $r_1 = 1$, which implies that by equation (4c), $m = 1$. Thus, by equation (4a), it produced

$$g(y; u, v, w) = (vy + w) - (y + u)$$

Observe that
$$g(y; u, v, w) = 0;$$

$$\Rightarrow (vy + w) - (y + u) = 0$$

$$\Rightarrow (v - 1)y + (w - u) = 0 \tag{5}$$

$$\Rightarrow y = \frac{-(w - u)}{v - 1} \tag{6}$$

Is the seeking solution to the zero of the auxiliary function $g(.)$.

The depressed equation $p^*_{1,1} y = 0$ implies that

$$y = 0 \tag{7}$$

To determine the unknown parameters, for completeness of procedure, the equation (5) and equation (7) are compared to produce

$$v - 1 = 1$$

$$w - u = 0 \implies u = w$$

Substituting these values into equation (6) producing

$$y = \frac{u - w}{v - 1} = 0$$

So that by equation (4b) and Lemma 2.3, it produced

$$x = \beta_1 = -\frac{a_1}{a_2}$$

Which is a general solution to the linear equation.

3.2 The Quadratic Case

It is clear that when $n = 2$, then $f_2(x) = f_2(\beta_2 + y) = 0$ implies that $\sum_{\substack{k=0 \\ k \neq 1}}^{2} p_{2,k}^* y^k = 0$. It may simply write this as

$$y^2 + p_{2,0}^* = 0 \tag{8}$$

Now, for $n = 2$ there is at least one $r_2 \in [2] = \{1,2\}$ such that $mr_2 = 2$. It suffices to take $r_2 = 2$ which implies that $m = 1$. Thus, by equation (4a) produced

$$g(y; u, v, w) = (vy + w) - (y^2 + u)$$

Observe that

$$g(y; u, v, w) = 0;$$

$$\implies y^2 - vy + u - w = 0 \tag{9}$$

$$\implies y = \frac{v \pm \sqrt{v^2 - 4(u - w)}}{2} \tag{10}$$

This is called as the seeking solution for the depressed quadratic equation.

Now comparing the depressed quadratic equation (8) and equation (9) to determine the unknown parameters u, v and w, it is obtained

$$v = 0 \; and \; u - w = p_{2,0}^* \tag{11}$$

Substituting these values into equation (10) gives

$$y = \pm \sqrt{-p_{2,0}^*}$$

So that by equation (4b) and Lemma 2.3 are producing

$$= \beta_2 \pm \sqrt{-p_{2,0}^*} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2} \tag{12}$$

is a general solution to the quadratic equation.

3.3 The Cubic Case

Also, observe that when $n = 3$, then $f_3(x) = f_3(\beta_3 + y) = 0$ implies that $\sum_{\substack{k=0 \\ k \neq 2}}^{3} p_{3,k}^* y^k = 0$. Hence, this is written as

$$p_{3,0}^* + p_{3,1}^* y + y^2 = 0 \tag{13}$$

Now, since $n = 3$, there is at least one $r_3 \in [3] = \{1,2,3\}$ such that $mr_3 = 3$. It suffices to take $r_3 = 1$ which implies that $m = 3$. Thus, by equation (4a), it produced

$$g(y; u, v, w) = (vy + w)^3 - (y + u)^3$$

and

$$g(y; u, v, w) = 0;$$

$$\Longrightarrow \quad (vy + w)^3 - (y + u)^3 = 0 \tag{14}$$

$$\Longrightarrow (vy + w)^3 = (y + u)^3; \ \Longrightarrow \ vy + w = y + u; \ \Longrightarrow$$

$$y = \frac{u - w}{v - 1} \tag{15}$$

In order to solve equation (13), the equation (14) is expanded and obtained

$$(v^3 - 1)y^3 + 3(v^2 w - u)y^2 + 3(vw^2 - u^2)y + w^3 - u^3 = 0 \tag{16}$$

Now, comparing the coefficients in the two equations (16) and (13), it is obtained

$$v^3 - 1 = 1 \tag{17a}$$

$$v^2 w - u = 0 \tag{17b}$$

$$vw^2 - u^2 = \frac{p_{3,1}^*}{3} \tag{17c}$$

$$w^3 - u^3 = p_{3,0}^* \tag{17d}$$

In equation (17b), it follows that $u = v^2 w$. Substituting into (17c), (17d) and using (17a) producing

$$vw^2 = -\frac{p_{3,1}^*}{3} \tag{18}$$

$$(v^3 + 1)w^3 = -p_{3,0}^* \tag{19}$$

From equation $(17c) = -\frac{p_{3,1}^*}{3w^2}$, substituting this into equation (17d) and taking $t = w^3$, it produced

$$27t^2 + 27p_{3,0}^* t - p_{3,1}^{*}{}^3 = 0 \tag{20}$$

is a quadratic equation in t whose solution is

$$t_j = -\left(\frac{p_{3,0}^*}{2}\right) + (-1)^j \sqrt{\left(\frac{p_{3,0}^*}{2}\right)^2 + \left(\frac{p_{3,1}^*}{3}\right)^3} \ ; j = 1,2 \tag{21}$$

It follows that

$$w_j = \sqrt[3]{-\left(\frac{p_{3,0}^*}{2}\right) + (-1)^j \sqrt{\left(\frac{p_{3,0}^*}{2}\right)^2 + \left(\frac{p_{3,1}^*}{3}\right)^3}} \quad ; j = 1,2 \tag{22}$$

From the equations: (15), (17a) and (17b), the $y = \frac{u-w}{v-1}, v = -\frac{p_{3,1}^*}{3w^2}$ and $u = v^2 w$ are produced which implies that $y = \left(w - \frac{p_{3,1}^*}{3} w^{-1}\right)$. Thus, altogether, it becomes

$$y_{1j} = \left(-\left(\frac{p_{3,0}^*}{2}\right) + (-1)^j \sqrt{\left(\frac{p_{3,0}^*}{2}\right)^2 + \left(\frac{p_{3,1}^*}{3}\right)^3}\right)^{\frac{1}{3}}$$

$$- \frac{p_{3,1}^*}{3}\left(-\left(\frac{p_{3,0}^*}{2}\right) + (-1)^j \sqrt{\left(\frac{p_{3,0}^*}{2}\right)^2 + \left(\frac{p_{3,1}^*}{3}\right)^3}\right)^{-\frac{1}{3}} \quad ; \quad j = 1,2 \tag{23}$$

Simply put for every $j = 1, 2$

$$y_{1j} = \sum_{m=0}^{1} R_m : R_m = \left(-\frac{p_{3,1}^*}{3}\right)^m \left(-\frac{p_{3,0}^*}{2} + (-1)^j \sqrt{\left(\frac{p_{3,0}^*}{2}\right)^2 + \left(\frac{p_{3,1}^*}{3}\right)^3}\right)^{\frac{1}{3}(-1)^m}$$

Is a solution to equation (16). So that by equation (4b) it follows that for each y_{1j} $(j = 1,2)$

$$x_{1j} = \sum_{m=0}^{1} R_m + \beta_3 \quad ; j = 1,2 \tag{24}$$

is a solution to the cubic equation. However, for each j $(j = 1,2)$, x_{1j} is one of the three roots and the remaining roots x_{2j}, x_{3j} can be found by employing Vieta identity formula for relationship between roots given by:

$$y_{1j} + y_{2j} + y_{3j} = 0, \ y_{1j}y_{2j} + y_{2j}y_{3j} + y_{3j}y_{1j} = p_{3,1}^*$$

From these, every fixed j a quadratic equation is obtained whose solution gives

$$y_{ij} = \left(\frac{-y_{1j}}{2}\right) + (-1)^i \sqrt{-\left(\frac{3}{4}\right)y_{1j}^2 - p_{3,1}^*}; i = 2,3 \tag{25}$$

Equation (4b) and lemma 2.3 completely define the solution of the general cubic equation. Hence, for each fixed j $(j = 1,2)$, it obtained x_{ij} $(i = 1,2,3)$. Thus, it is shown that $\forall j \in \{1, 2\}$, there exist y_{ij} (solution to equation (16)) such that by equation (4b) and Lemma 2.3 producing

$$x_{ij} = y_{ij} + \beta_3 \ ; i = 1,2,3$$

is a solution to the general cubic equation.

3.4   The Quartic Case

It is clear that when $n = 4$, then $f_4(x) = f_4(\beta_4 + y) = 0$ implies that $\sum_{\substack{k=0 \\ k \neq 3}}^{4} p_{4,k}^* y^k = 0$. It is simply written as

$$p_{4,0}^* + p_{4,1}^* y + p_{4,2}^* y^2 + y^4 = 0 \tag{26}$$

For $n = 4$, there is at least one $r_4 \in [4] = \{1,2,3,4\}$ such that $mr_4 = 4$. It suffices to take $r_4 = 2$ which implies that $m = 2$. So that

$$g(y; u, v, w) = (vy + w)^2 - (y^2 + u)^2$$

In order to solve equation (27), first it considers the solution to

$$g(y; u, v, w) = 0;$$

$$\Rightarrow (vy + w)^2 - (y^2 + u)^2 = 0 \tag{27}$$

$$\Rightarrow (y^2 + u)^2 = (vy + w)^2; \Rightarrow y^2 + u = \pm(vy + w);$$

$$\Rightarrow y^2 + u = (vy + w) \text{ or } y^2 + u = -(vy + w);$$

$$\Rightarrow y^2 - vy + (u - w) = 0 \text{ or } y^2 + vy + (u + w) = 0;$$

$$\Rightarrow y_1 = \frac{v - \sqrt{v^2 - 4(u - w)}}{2} \text{ or } y_2 = \frac{v + \sqrt{v^2 - 4(u - w)}}{2} \tag{28}$$

$$\Rightarrow y_3 = \frac{-v - \sqrt{v^2 - 4(u + w)}}{2} \text{ or } y_4 = \frac{-v + \sqrt{v^2 - 4(u + w)}}{2} \tag{29}$$

Constitute the seeking solution to the quartic equation (26). Now expanding equation (27), it obtained

$$y^4 + (2u - v^2)y^2 - 2wvy + u^2 - w^2 = 0 \tag{30}$$

Now, comparing the coefficients in the two equations (27) and (30) producing

$$p_{4,2}^* = (2u - v^2) \Rightarrow u = \frac{p_{4,2}^* + v^2}{2} \tag{31}$$

$$p_{4,1}^* = -2wv ; \Rightarrow w = \frac{-p_{4,1}^*}{2v} \tag{32}$$

$$p_{4,0}^* = (u^2 - w^2) ; \Rightarrow p_{4,0}^* = \left(\frac{p_{4,2}^* + v^2}{2}\right)^2 - \left(\frac{-p_{4,1}^*}{2v}\right)^2 \tag{33}$$

The equation (33) is expanded to have

$$v^6 + 2p_{4,2}^* v^4 + \left(p_{4,2}^{*2} - 4p_{4,0}^*\right)v^2 - p_{4,1}^{*2} = 0 \tag{34}$$

Which is the resolvent equation associated with the quartic equation.

Let

$$v^2 = z \tag{35}$$

Then, by substituting into equation (34), it produced

$$z^3 + 2p_{4,2}^* z^2 + \left(p_{4,2}^{*2} - 4p_{4,0}^*\right)z - p_{4,1}^{*2} = 0$$

Thus, using any standard method for solving cubic equation it obtained at least one solution in z, consequently v using the relating equation (35).

Now, recall from the equations (31) and (32) that $u = \frac{p_{4,2}^* + v^2}{2}$ and $w = \frac{-p_{4,1}^*}{2v}$ which implies that

$$u + w = \frac{v\left(p_{4,2}^* + v^2\right) - p_{4,1}^*}{2v} \quad and \quad u - w = \frac{v\left(p_{4,2}^* + v^2\right) + p_{4,1}^*}{2v}$$

Then the above seeking solution to equation (26) becomes

$$y_1 = \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v\left(p_{4,2}^* + v^2\right) + p_{4,1}^*}{2v}} \quad or \quad y_2 = \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v\left(p_{4,2}^* + v^2\right) + p_{4,1}^*}{2v}}$$

$$y_3 = -\frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v\left(p_{4,2}^* + v^2\right) - p_{4,1}^*}{2v}} \quad or \quad y_4 = -\frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v\left(p_{4,2}^* + v^2\right) - p_{4,1}^*}{2v}}$$

Consequently, the solution to the general quartic becomes

$$x_1 = \frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v\left(p_{4,2}^* + v^2\right) + p_{4,1}^*}{2v}} + \beta_4 \quad or \quad x_2 = \frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v\left(p_{4,2}^* + v^2\right) + p_{4,1}^*}{2v}} + \beta_4$$

$$x_3 = -\frac{v}{2} - \sqrt{\frac{v^2}{4} - \frac{v\left(p_{4,2}^* + v^2\right) - p_{4,1}^*}{2v}} + \beta_4 \quad or \quad x_4 = -\frac{v}{2} + \sqrt{\frac{v^2}{4} - \frac{v\left(p_{4,2}^* + v^2\right) - p_{4,1}^*}{2v}} + \beta_4$$

Alternatively, this can be simply put as

$$x_{k,j} = \frac{(-1)^k v^2 + (-1)^j \sqrt{v^4 - 2v\left(v\left(v^2 + p_{4,2}^*\right) + (-1)^k p_{4,1}^*\right)}}{2v} - \frac{a_3}{4a_4}; k, j = 1,2 \tag{36}$$

It now remains to demonstration with telling examples the validity of our results for solving $f_n(x) = 0$ for $n \in N$ such that $n < 5$. Clearly for $n = 1$ and $n = 2$ is quite trivial, thus it shall justify the results for the cases $n = 3$ and $n = 4$.

3.5   Applications

It now remains to apply and demonstration with telling examples the validity of the results for solving $f_n(x) = 0$ for every $n \in N$ such that $n < 5$. Clearly for $n = 1$ and $n = 2$ is quite trivial, hence the examples that follows justifies our results for the cases $n = 3$ and $n = 4$.

Example 1: Find the value of x given that: $x^3 - x^2 - 5x - 3 = 0$.

Solution

Using equation (23),

$$y_{1j} = \left(-\frac{p_{3,0}^*}{2} + (-1)^j \sqrt{\left(\frac{p_{3,0}^*}{2}\right)^2 + \left(\frac{p_{3,1}^*}{3}\right)^3}\right)^{\frac{1}{3}} - \frac{p_{3,1}^*}{3}\left(-\frac{p_{3,0}^*}{2} + (-1)^j \sqrt{\left(\frac{p_{3,0}^*}{2}\right)^2 + \left(\frac{p_{3,1}^*}{3}\right)^3}\right)^{-\frac{1}{3}}$$

where $p^*_{n,k} = \sum_{j=0}^{n-k} \frac{(-1)^j}{a_n} \binom{k+j}{j} a_{k+j} \left(\frac{a_{n-1}}{na_n}\right)^j$; $a_3 = 1$, $a_2 = -1$, $a_1 = -5$, $a_0 = -3$

Then,

$$p^*_{3,1} = \sum_{j=0}^{3-1} \frac{(-1)^j}{a_3} \binom{1+j}{j} a_{1+j} \left(\frac{a_2}{3a_3}\right)^j = \frac{1}{1}\sum_{j=0}^{2} \binom{1+j}{j} a_{1+j} \left(\frac{-1}{3\times1}\right)^j = \frac{-16}{3}$$

$$p^*_{3,0} = \sum_{j=0}^{3-0} \frac{(-1)^j}{a_3} \binom{0+j}{j} a_{1+j} \left(\frac{a_2}{3a_3}\right)^j = \frac{1}{1}\sum_{j=0}^{3} a_{1+j} \left(\frac{-1}{3\times1}\right)^j = \frac{-128}{27}$$

$$y_{1j} = \left(\frac{-\left(\frac{-128}{27}\right)}{2} + (-1)^j \sqrt{\frac{\left(\frac{-128}{27}\right)^2}{4} + \frac{\left(\frac{-16}{3}\right)^3}{27}}\right)^{\frac{1}{3}} - \left(\frac{-16}{3}\right) \left(\frac{-\left(\frac{-128}{27}\right)}{2} + (-1)^j \sqrt{\frac{\left(\frac{-128}{27}\right)^2}{4} + \frac{\left(\frac{-16}{3}\right)^3}{27}}\right)^{-\frac{1}{3}}$$

$$= \left(\frac{128}{54} + (-1)^j \sqrt{\frac{16384}{2916} - \frac{4096}{729}}\right)^{\frac{1}{3}} + \left(\frac{16}{9}\right)\left(\frac{128}{54} + (-1)^j \sqrt{\frac{16384}{2916} - \frac{4096}{729}}\right)^{-\frac{1}{3}}$$

$$= \left(\frac{128}{54} + (-1)^j \sqrt{\frac{16384-16384}{2916}}\right)^{\frac{1}{3}} + \left(\frac{16}{9}\right)\left(\frac{128}{54} + (-1)^j \sqrt{\frac{16384-16384}{2916}}\right)^{-\frac{1}{3}} = \frac{8}{3} \text{ (twice)}$$

Thus,

$$x_{1j} = y_{1j} - \frac{a_2}{3a_3} = \frac{8}{3} - \frac{-1}{3\times1} = 3$$

the remaining roots x_{2j}, x_{3j} can be found by employing

$$y_{kj} = \left(\frac{-y_{1j}}{2}\right) + (-1)^k \sqrt{-\left(\frac{3}{4}\right)y_{1j}^2 - p^*_{3,1}} \; ; \; k = 2,3$$

$$= \left(\frac{-\frac{8}{3}}{2}\right) + (-1)^k \sqrt{-\left(\frac{3}{4}\right)\left(\frac{8}{3}\right)^2 - \frac{-16}{3}} = \left(\frac{-\frac{8}{3}}{2}\right) + (-1)^k \sqrt{-\frac{16}{3} + \frac{16}{3}} = -\frac{4}{3} \text{ (twice)}$$

Hence

$$x_{kj} = y_{kj} - \frac{a_2}{3a_3} = -\frac{4}{3} - \frac{-1}{3\times1} = -1 \text{ (twice)}$$

$$\Rightarrow \forall j = 1,2 ; \; x_{kj} = 3, -1, -1 \; ; k = 1,2,3 \text{ respectively.}$$

Example 2: Find the roots of

$$x^4 + 4x^3 + 33x^2 + (58 - 14i)x + (148 - 14i) = 0$$

Solution

Using equation (36)

$$x_{k,j} = \frac{(-1)^k v^2 + (-1)^j \sqrt{v^4 - 2v(v(v^2 + p^*_{4,2}) + (-1)^k p^*_{4,1})}}{2v} - \frac{a_3}{4a_4}; k = 1, 2; j = 1, 2$$

and the associated resolvent cubic equation

$$z^3 + 2p^*_{4,2}z^2 + \left(p^*_{4,2}{}^2 - 4p^*_{4,0}\right)z - p^*_{4,1}{}^2 = 0 : v^2 = z$$

where $p^*_{n,k} = \sum_{j=0}^{n-k} \frac{(-1)^j}{a_n} \binom{k+j}{j} a_{k+j} \left(\frac{a_{n-1}}{na_n}\right)^j$;

$$a_4 = 1, a_3 = 4, a_2 = 33, a_1 = (58 - 14i), a_0 = (148 - 14i)$$

Then,

$$pss^*_{4,2} = \sum_{j=0}^{4-2} \frac{(-1)^j}{a_4} \binom{2+j}{j} a_{2+j}(\beta_4)^j = \sum_{j=0}^{2} \frac{(-1)^j}{1} \binom{2+j}{j} a_{2+j} \left(\frac{4}{4 \times 1}\right)^j = 27$$

$$p^*_{4,1} = \sum_{j=0}^{4-1} \frac{(-1)^j}{a_4} \binom{1+j}{j} a_{1+j}(\beta_4)^j = \sum_{j=0}^{3} \frac{(-1)^j}{1} \binom{1+j}{j} a_{1+j} \left(\frac{4}{4 \times 1}\right)^j = -14i$$

$$p^*_{4,0} = \sum_{j=0}^{4-0} \frac{(-1)^j}{a_4} \binom{0+j}{j} a_{0+j}(\beta_4)^j = \sum_{j=0}^{4} \frac{(-1)^j}{1} a_j \left(\frac{4}{4 \times 1}\right)^j = 120$$

Thus

$$x_{k,j} = \frac{(-1)^k v^2 + (-1)^j \sqrt{v^4 - 2v(v(v^2 + 27) - (-1)^k 14i)}}{2v} - \frac{4}{4 \times 1}; k = 1, 2; j = 1, 2$$

To solve for v, the resolvent cubic equation is used; that is

$$z^3 + 2p^*_{4,2}z^2 + \left(p^*_{4,2}{}^2 - 4p^*_{4,0}\right)z - p^*_{4,1}{}^2 = 0 : v^2 = z$$

where $p^*_{4,2} = 27, p^*_{4,1} = -14i, \ p^*_{4,0} = 120$ substitute this value producing

$$z^3 + 54z^2 + 249z + 196 = 0$$

Now, on re-applying the method of example 1 above to obtain the first root of the resolvent cubic equation, hence $z = -1$ is one of such roots. Since $v^2 = z \Rightarrow v = \pm i$.

Using $v = i$, it can be seen that

$$x_{k,j} = \frac{(-1)^k i^2 + (-1)^j \sqrt{i^4 - 2i(i(i^2 + 27) - (-1)^k 14i)}}{2i} - 1; k = 1, 2; j = 1, 2$$

Hence the roots are $x_{k,j} = -1 + 3i, -1 - 2i, -1 + 4i, -1 - 5i$; $k = 1, 2; j = 1, 2$

Note that if $v = -i$ is used, the same values for the roots $x_{k,j}$ ($k = 1, 2; j = 1, 2$) above is obtained.

4.   Conclusion

In this study, a function in terms of combinatorial coefficient that compute the coefficients of depressed polynomial equations and then apply the same to obtain solution to a polynomial equation of degree less than five in one variable is derived. In particular, the result of the work is a unification and improvement from previous authors which are only applicable for the case of polynomial equation of degree one. Furthermore, the results of this study improve and generalize the result recently announced by Tiruneh (2019, 2020). For further research, it is recommended that the effort should be made toward providing other variant methods that are simpler and friendly.

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