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ON A NEW GENERALIZED BETA FUNCTION DEFINED BY THE GENERALIZED WRIGHT FUNCTION AND ITS APPLICATIONS

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ABSTRACT

Various extensions of classical gamma, beta, Gauss hypergeometric and confluent hypergeometric functions have been proposed recently by many researchers. In this paper, further generalized extended beta function with some of its properties like summation formulas, Integral representations, connections with other special functions such as incomplete gamma, classical beta, classical Wright, hypergeometric, error, Fox-H, Fox-Wright, Meijer-G functions are obtained. Beta distribution together with-it corresponding moment, mean, variance, moment generating function and cumulative distribution are also presented. Moreover, the generalized beta function is used to generalized classical and other related extended Gauss, Kumar confluent, Appell's and Lauricella's hypergeometric functions with their integral representations, differential, difference, summation and transformations formulas.

Keywords: *Appell-Lauricella function, Beta distribution, Confluent hypergeometric function, Gauss hypergeometric function, Wright function.*

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1. Introduction

The classical gamma and beta functions are given in (Chaudhry & Zubair, 2002; Srivastava & Choi, 2012)

$$\Gamma(y_1)=\int_0^\infty y^{y_1-1}\exp(-y)\,dy,\quad (\mathrm{Re}(y_1)>0), \tag{1}$$

$$B(\mathrm{y}_1,y_2)=\int_0^1 y^{y_1-1}(1-y)^{y_2-1}\,dt,\qquad (\mathrm{Re}(y_1)>0,\mathrm{Re}(y_2)>0). \tag{2}$$

Classical Gauss and Kumar confluent hypergeometric functions are defined as (see for example, Abubakar, 2021a)

$$F(\mathrm{g}_1,\mathrm{g}_2;\mathrm{g}_3;z)=\sum_{u=0}^{\infty}(g_1)_u\frac{B(g_2+u,\mathrm{g}_3-g_2)}{B(g_2,\mathrm{g}_3-g_2)}\frac{z^u}{\mathrm{u}!}, \tag{3}$$

$$\left(|z|<1;\operatorname{Re}(g_3)>\operatorname{Re}(g_2)>0\right).$$

&

$$\Phi(g_2;g_3;z)=\sum_{u=0}^{\infty}\frac{B(g_2+u,g_3-g_2)}{B(g_2,g_3-g_2)}\frac{z^u}{u!}, \tag{4}$$

$$\left(\operatorname{Re}(g_3)>\operatorname{Re}(g_2)>0\right).$$

Chaudhry & Zubair (1994) generalized classical gamma function in equation (1) as follow:

$$\Gamma_{q_1}(y_1)=\int_0^{\infty} y^{y_1-1}\exp\left(-y-\frac{q_1}{y}\right)dy, \quad \left(\operatorname{Re}(y_1)>0,\operatorname{Re}(y_2)>0\right). \tag{5}$$

Chaudhry *et al.*, (1997) generalized classical beta function in equation (2) as follow:

$$B_{q_1}(y_1,y_2)=\int_0^1 y^{y_1-1}(1-y)^{y_2-1}\exp\left(-\frac{q_1}{(1-y)y}\right)dy, \tag{6}$$

$$\left(\operatorname{Re}(y_1)>0,\operatorname{Re}(y_2)>0,\operatorname{Re}(q_1)>0\right).$$

Chaudhry *et al.* (1997) showed that the extension of beta function has certain connection with Macdonal, error and Whittaker functions.

Chaudhry *et al.* (2004) generalized classical Gauss and Kumar confluent hypergeometric functions in equations (3) and (4) using extended Euler's beta function in (6) as follows:

$$F_{q_1}(g_1,g_2;g_3;z)=\sum_{u=0}^{\infty}(g_1)_u\frac{B_{q_1}(g_2+u,g_3-g_2)}{B(g_2,g_3-g_2)}\frac{z^u}{u!}, \tag{7}$$

$$\left(q_1\geq 0,|z|<1;\operatorname{Re}(g_3)>\operatorname{Re}(g_2)>0\right).$$

and

$$\Phi_{q_1}(g_2;g_3;z)=\sum_{u=0}^{\infty}\frac{B_{q_1}(g_2+u,g_3-g_2)}{B(g_2,g_3-g_2)}\frac{z^u}{u!}, \tag{8}$$

$$\left(q_1\geq 0,\operatorname{Re}(g_3)>\operatorname{Re}(g_2)>0\right).$$

Choi *et al.* (2014) proposed extension of the extended Euler beta function (6) as shown below:

$$B_{q_1,q_2}(y_1,y_2)=\int_0^1 y^{y_1-1}(1-y)^{y_2-1}\exp\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy, \tag{9}$$

$$\left(\operatorname{Re}(y_1)>0,\operatorname{Re}(y_2)>0,\operatorname{Re}(q_1)>0,\operatorname{Re}(q_2)>0\right).$$

Choi *et al.* (2014) showed that the extension of beta function has certain connections with Laguree polynomial, Macdonal and Whittaker functions. Ata (2018) presented extension of the extended gamma and beta functions in (5) and (6) as follows:

$$ {}^{\psi}\Gamma_{q_1}^{(\alpha,\beta)}\left(y_1\right)=\int_0^\infty t^{y_1-1}\,{}_1\psi_1\left(\alpha,\beta;-y-\frac{q_1}{y}\right)dy, \tag{10}$$

$$\left(\operatorname{Re}\left(y_1\right)>0,\operatorname{Re}\left(\alpha\right)>0,\operatorname{Re}\left(\beta\right)>1,\operatorname{Re}\left(q_1\right)>0\right).$$

and

$$ {}^{\psi}B_{q_1}^{(\alpha,\beta)}\left(\mathrm{y}_1,y_2\right)=\int_0^1 y^{y_1-1}\left(1-y\right)^{y_2-1}\,{}_1\psi_1\left(\alpha,\beta;-\frac{q_1}{\left(1-y\right)y}\right)dy, \tag{11}$$

$$\left(\operatorname{Re}\left(y_1\right)>0,\operatorname{Re}\left(y_2\right)>0,\operatorname{Re}\left(\alpha\right)>0,\operatorname{Re}\left(\beta\right)>1,\operatorname{Re}\left(q_1\right)>0\right).$$

where ${}_1\psi_1(;)$ is classical Wright function defined in (Rainville, 1960).

Kulip *et al.* (2020) proposed further extension of the extended gamma and beta functions in (9) and (10) as follows:

$$ {}^{W}\Gamma_{q_1}^{(\alpha,\beta;\tau,\delta)}\left(y_1\right)=\int_0^\infty y^{y_1-1}W_{\alpha,\beta}^{\tau,\delta}\left(-y-\frac{q_1}{y}\right)dy, \tag{12}$$

$$\left(\operatorname{Re}\left(y_1\right)>0,\alpha,\beta,\tau,\delta\in C;\alpha>-1,\delta\notin Z_0^-,\mathrm{q}_1\geq 0\right).$$

and

$$ {}^{W}B_{q_1}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,y_2\right)=\int_0^1 y^{y_1-1}\left(1-y\right)^{y_2-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{\left(1-y\right)y}\right)dy, \tag{13}$$

$$\left(\operatorname{Re}\left(y_1\right)>0,\operatorname{Re}\left(y_2\right),\alpha,\beta,\tau,\delta\in C;\alpha>-1,\delta\notin Z_0^-,\mathrm{q}_1\geq 0\right),$$

where $W_{\alpha,\beta}^{\tau,\delta}(;)$ is the generalized Wright function defined by El-Shahed & Salem (2015).

Other generalizations of gamma, beta, Gauss hypergeometric, confluent hypergeometric and other related functions with their properties can be found in (Ibrahim & Ozel, 2016; Srivastava *et al.*, 2017; Abdulnaby & Ibrahim, 2020; Abubakar & Kabara, 2019a, b; Abubakar *et al.*, 2020; Ata & Kiymaz, 2020; Emmanuel, 2020; Gehlot & Nisar, 2020 & Ghayasuddin *et al.*, 2020; Abubakar, 2021b). It is the purpose of this article to introduce the following new generalization of Euler beta function:

$$ {}^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,y_2\right)=\int_0^1 y^{y_1-1}\left(1-y\right)^{y_2-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy\,, \tag{14}$$

$$\left(\mathrm{Re}\left(y_1\right)>0,\mathrm{Re}\left(y_2\right)>0,\alpha,\beta,\tau,\delta\in C;\alpha>-1,\delta\notin Z_0^-,\mathrm{q}_1\geq 0,q_2\geq 0\right).$$

2. Generalized Beta Function

In this section, summation formulas and symmetric relation of generalized beta function are discussed in the following theorems:

Theorem 2.1: For the generalized beta function, the following holds true:

$$^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,y_2\right)={}^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,y_2+1\right)+{}^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(y_1+1,y_2\right).$$

Proof: By direct calculation,

$$^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,y_2\right)=\int_0^1 y^{y_1}\left(1-y\right)^{y_2}\left[y^{-1}+\left(1-y\right)^{-1}\right]W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy.$$

On simplifying, gives

$$^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,y_2\right)={}^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,y_2+1\right)+{}^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(y_1+1,y_2\right).$$

Theorem 2.2: The following summation formulas hold:

$$^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,1-y_2\right)=\sum_{u=0}^{\infty}\frac{\left(y_2\right)_u}{\mathrm{u}!}\,{}^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(y_1+u,1\right).$$

Proof: By direct computation, yields

$$^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,1-y_2\right)=\int_0^1 y^{y_1-1}\sum_{u=0}^{\infty}\frac{\left(y_2\right)_u y^u}{\mathrm{u}!}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy.$$

Reversing the order of summation and integration, gives

$$^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,1-y_2\right)=\sum_{u=0}^{\infty}\frac{\left(y_2\right)_u}{\mathrm{u}!}\int_0^1 y^{y_1+u-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy.$$

On simplifying, yields

$$^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,1-y_2\right)=\sum_{u=0}^{\infty}\frac{\left(y_2\right)_u}{\mathrm{u}!}\,{}^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(y_1+u,1\right).$$

Theorem 2.3: The following summation formula holds:

$$^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(\mathrm{y}_1,y_2\right)=\sum_{u=0}^{\infty}{}^{W}B_{\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(y_1+u,y_2+1\right).$$

Proof: Using equation (14), it is easy to have

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=\int_0^1 y^{y_1-1}(1-y)^{y_2}\sum_{u=0}^{\infty}y^u\, W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy. $$

Reversing the order of summation and integration, gives

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=\sum_{u=0}^{\infty}\int_0^1 y^{y_1+u-1}(1-y)^{y_2}\, W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy. $$

On simplifying, leads to the following

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=\sum_{u=0}^{\infty}{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1+u,y_2+1). $$

Corollary 2.1: The following summation formula hold true:

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=\sum_{u=0}^{n}\binom{n}{u}{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1+u,y_2+n-u). $$

The next theorem represents the integral representations generalized beta function.

Theorem 2.4: Each of the following integral representations hold true:

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=n\int_0^1 y^{ny_1-1}\left(1-y^n\right)^{y_2-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y^n}-\frac{q_2}{1-y^n}\right)dy, \tag{15} $$

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=2^{1-y_1-y_2}\int_{-1}^1 (1+y)^{y_1-1}(1-y)^{y_2-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{2q_1}{1+y}-\frac{2q_2}{1-y}\right)dy, \tag{16} $$

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=a^{1-y_1-y_2}\int_0^a y^{y_1-1}(a-y)^{y_2-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{aq_1}{y}-\frac{aq_2}{a-y}\right)dt, \tag{17} $$

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=a^{y_2}(1+a)^{y_1}\int_0^1 \frac{y^{y_1-1}(1-y)^{y_2-1}}{(1+y)^{y_1+y_2}}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1(y+a)}{(a+1)y}-\frac{q_2(y+a)}{a(1-y)}\right)dy. \tag{18} $$

Proof: Substituting $y=r^n$, $y=2^{-1}(1+r)$, $y=a^{-1}r$ and $y=(1+a)(r+a)^{-1}r$ in equation (14), gives the desired results in (15), (16), (17), (18) and (19), respectively.

Theorem 2.5: The following integral representation hold true.

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=2\int_0^{\frac{\pi}{2}}\cos^{2y_1-1}\theta\sin^{2y_2-1}\theta\, W_{\alpha,\beta}^{\tau,\delta}\left(-q_1\sec^2\theta-q_2\cos ec^2\theta\right)d\theta. \tag{19} $$

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=\int_0^\infty \frac{y^{y_1-1}}{(1+y)^{y_1+y_2}}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1(1+y)}{y}-q_2(1+y)\right)dy, \tag{20} $$

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=(c-a)^{1-y_1-y_2}\int_a^c (y-a)^{y_1-1}(c-y)^{y_2-1} $$

$$ \times W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1(c-a)}{(y-a)}-\frac{q_2(c-a)}{(c-y)}\right)dy. \tag{21} $$

Proof: Setting $y=cos^2\theta,\ y=r(1+r)^{-1}\ and\ \ y=(r-a)(c-a)^{-1}$ into equation (14), lead to the required results in (19), (20) and (21), respectively.

Theorem 2.6: The following summation formulas hold true.

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\tau+u+1)}(y_1,y_2)=(\tau)_{u+1}\sum_{r=0}^{n}\frac{(-1)^u}{u!(n-u)!(\gamma+u)}\,{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau+u,\tau+u+1)}(y_1,y_2). \tag{22} $$

Proof: This result can be obtained using the results in (El-Shahed & Salem, 2015; Kulib *et al.*, 2020)

$$ W_{\alpha,\beta}^{\tau,\tau+u+1}(z)=(\tau)_{u+1}\sum_{u=0}^{n}\frac{(-1)^u}{u!(n-u)!(\tau+u)}W_{\alpha,\beta}^{\tau+u,\tau+u+1}(z). $$

For u = 1, in equation (22), obtained the following recurrence relation.

Corollary 2.2: The following relation hold:

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\tau+2)}(y_1,y_2)=(1+\gamma)\ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\tau+1)}(y_1,y_2)+\tau\ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau+1,\tau+2)}(y_1,y_2). $$

Theorem 2.7: The following formula hold:

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta-1;\tau,\delta)}(y_1,y_2)+(1-\beta)\ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=-\frac{\alpha\tau q_1}{\delta}\ {}^{W}B_{q_1,q_2}^{(\alpha,\alpha+\beta;\tau+1,\delta+1)}(y_1-1,y_2-1) $$

$$ -\frac{\alpha\tau(q_2-q_1)}{\delta}\ {}^{W}B_{q_1,q_2}^{(\alpha,\alpha+\beta;\tau+1,\delta+1)}(y_1,y_2-1). $$

Proof: This result can be obtained using (El-Shahed & Salem, 2015; Kulib *et al.*, 2020)

$$ W_{\alpha,\beta-1}^{\tau,\delta}(z)+(1-\beta)\,W_{\alpha,\beta}^{\tau,\delta}(z)=\frac{\alpha\tau z}{\delta}\,W_{\alpha,\alpha+\beta}^{\tau+1,\delta+1}(z). $$

Connection between the introduced beta function and incomplete gamma, classical beta, classical Wright, hypergeometric, error, Fox-H, Fox-Wright, Meijer G-, Mittag-Leffler functions is discuss in the following theorems:

Theorem 2.8: Each of the following integral representations hold true.

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=\frac{\Gamma(\delta)}{\Gamma(\tau)}\int_0^1 y^{y_1-1}(1-y)^{y_2-1}H_{13}^{11}\left[\frac{q_1}{y}+\frac{q_2}{1-y}\middle|\begin{matrix}(1-\tau,1)\\(0,1),(1-\beta,\alpha),(1-\delta,1)\end{matrix}\right]dy, \quad (23) $$

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=\frac{\Gamma(\delta)}{\Gamma(\tau)}\int_0^1 y^{y_1-1}(1-y)^{y_2-1}G_{13}^{11}\left[\frac{q_1}{y}+\frac{q_2}{1-y}\middle|\begin{matrix}1-\tau\\0,1-\beta,1-\tau\end{matrix}\right]dy, \quad (24) $$

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y)=\frac{\Gamma(\delta)}{\Gamma(\tau)}\int_0^1 y^{y_1-1}(1-y)^{y_2-1}\,{}_1\psi_2\left[\begin{matrix}(\tau,1) & ; \\ (\delta,1),(\beta,\alpha); & -\frac{q_1}{y}-\frac{q_2}{1-y}\end{matrix}\right]dy. \quad (25) $$

where $H_{q_1,q_2}^{m,n}(;)$, ${}_1\psi_2(;)$ and $G_{q_1,q_2}^{m,n}(;)$ represent Fox H-function, Fox-Wright function and Meijer G-function, respectively (see, for example, Andrew, 1985; Luchko, 2009; Mathai *et al.*, 2010; El-Shahed & Salem, 2015; Kulib *et al.*, 2020).

Proof: Equations (23), (24) and (25) can be obtained by using the relations in (El-Shahed & Salem, 2015; Kulib *et al.*, 2020)

$$ W_{\alpha,\beta}^{\tau,\delta}(-z)=\frac{\Gamma(\delta)}{\Gamma(\tau)}H_{13}^{11}\left[z\middle|\begin{matrix}(1-\tau,1)\\(0,1),(1-\beta,\alpha),(1-\delta,1)\end{matrix}\right], $$

$$ W_{\alpha,\beta}^{\gamma,\delta}(-z)=\frac{\Gamma(\delta)}{\Gamma(\tau)}G_{13}^{11}\left[z\middle|\begin{matrix}1-\tau\\0,1-\beta,1-\tau\end{matrix}\right], $$

$$ W_{\alpha,\beta}^{\gamma,\delta}(z)=\frac{\Gamma(\delta)}{\Gamma(\tau)}\,{}_1\psi_2\left[\begin{matrix}(\tau,1); & \\ (\delta,1),(\beta,\alpha); & z\end{matrix}\right], $$

respectively, the desired results are received.

Theorem 2.9: The following integral representations hold true.

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(y_1,y_2)=\frac{1}{B(\tau,\delta-\tau)}\int_0^1\int_0^1 u^{\tau-1}(1-u)^{\delta-\tau-1}y^{y_1}(1-y)^{y_2-1}W_{\alpha,\beta}\left(-\frac{q_1u}{y}-\frac{q_2u}{1-y}\right)dudy, $$

where $W_{\alpha,\beta}(;)$ denote well known classical Wright function defined in (Khan, et al., 2020).

Proof: This result can be obtained by using the following relation (El-Shahed & Salem, 2015; Kulib *et al.*, 2020)

$$ W_{\alpha,\beta}^{\tau,\delta}(z)=\frac{1}{B(\tau,\delta-\tau)}\int_0^1 u^{\tau-1}(1-u)^{\delta-\tau-1}W_{\alpha,\beta}(zu)\,du. $$

Theorem 2.10: Each of the following relation holds.

$$ {}^{W}B_{q_1,q_2}^{(0,\beta;1,\delta)}(y_1,y_2)=\frac{\Gamma(\delta)}{\Gamma(\beta)}\int_0^1 y^{y_1-1}(1-y)^{y_2-1}E_{1,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy, \quad (26) $$

$$ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;1,\delta)}(y_1,y_2)=\Gamma(\delta)\int_0^1 y^{y_1-1}(1-y)^{y_2-1}E_{(\alpha,1);(\beta,\delta)}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy, \quad (27) $$

$$ {}^{W}B_{q_1,q_2}^{\left(\frac{1}{2},1;\tau,\tau+1\right)}(y_1,y_2)=\frac{2^{\tau}}{\tau\sqrt{\pi}}\int_0^1 \frac{y^{y_1}(1-y)^{y_2}}{\left[(1-y)q_1+q_2y\right]}\gamma\left(\frac{1+\tau}{2},\left(\frac{q_1}{2y}+\frac{q_2}{2(1-y)}\right)^2\right)dy $$

$$ +\int_0^1 y^{y_1-1}(1-y)^{y_2-1}\,erf\left(\frac{q_1}{2y}+\frac{q_2}{2(1-y)}\right)dy. \quad (28) $$

where $E_{a,b}(;)$, erf $(;)$ and $\gamma\,(;)$ are two parameters Mittag-Leffler, error and incomplete gamma functions, respectively.

Proof: Equations (26), (27) and (28) can be obtained by using the following relation (El-Shahed & Salem, 2015)

$$ W_{0,\beta}^{1,\beta}(z)=\frac{\Gamma(\delta)}{\Gamma(\beta)}E_{1,\delta}(z), $$

$$ W_{\alpha,\beta}^{1,\delta}(z)=\Gamma(\delta)E_{(\alpha,1);(\beta,\delta)}(z), $$

$$ W_{-\frac{1}{2},\beta}^{\tau,\tau+1}(-z)=erf\left(\frac{z}{2}\right)+\frac{2^{\tau}}{\sqrt{\pi}z\tau}\gamma\left(\frac{1+\tau}{2},\frac{z^2}{4}\right), $$

respectively, the required results are obtained.

Theorem 2.11: Each of the following relation holds.

$$ {}^{W}B_{q_1,q_2}^{(0,\beta;\tau,\delta)}(y_1,y_2)=\frac{1}{\Gamma(\beta)}\int_0^1 y^{y_1-1}(1-y)^{y_2-1}\exp\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right) $$

$$ \times\,{}_1F_1\left(\delta-\tau;\delta;\frac{q_1}{y}+\frac{q_2}{1-y}\right)dy, \quad (29) $$

$$ {}^{W}B_{q_1,q_2}^{(1,\beta;\tau,\delta)}(y_1,y_2)=\frac{1}{\Gamma(\beta)}\int_0^1 y^{y_1-1}(1-y)^{y_2-1}\,{}_1F_2\left(\tau;\delta,\beta;\frac{q_1}{y}+\frac{q_2}{1-y}\right)dy, \quad (30) $$

$$ {}^{W}B_{q_1,q_2}^{\left(-\frac{1}{2},1;\tau,\delta\right)}\left(y_1,y_2\right)=-\frac{p\tau}{\delta\sqrt{\pi}}\int_0^1 {}_3F_3\left(\frac{1}{2},\frac{1+\tau}{2},\frac{2+\tau}{2};\frac{3}{2},\frac{1+\delta}{2},\frac{2+\delta}{2};-\left(-\frac{q_1}{2y}-\frac{q_2}{2(1-y)}\right)^2\right) $$

$$ \times y^{y_1-2}\left(1-y\right)^{y_2-1}-\frac{q\tau}{\delta\sqrt{\pi}}\int_0^1 {}_3F_3\left(\frac{1}{2},\frac{1+\tau}{2},\frac{2+\tau}{2};\frac{3}{2},\frac{1+\delta}{2},\frac{2+\delta}{2};-\left(-\frac{q_1}{2y}-\frac{q_2}{2(1-y)}\right)^2\right) $$

$$ \times y^{y_1-1}\left(1-y\right)^{y_2-2}dy+B\left(y_1,y_2\right). \qquad (31) $$

where ${}_qF_p(;)$ is the hypergeometric function and $B(y_1,y_2)$ is the classical beta function given in equation (2).

Proof: Equations (29), (30) and (31) can be obtained by using the following relation (El-Shahed & Salem, 2015)

$$ W_{\alpha,\beta}^{\tau,\delta}\left(z\right)=\frac{\exp\left(z\right)}{\Gamma\left(\beta\right)}\,{}_1F_1\left(\delta-\tau;\delta;-z\right), $$

$$ W_{1,\beta}^{\tau,\delta}\left(z\right)=\frac{1}{\Gamma\left(\beta\right)}\,{}_1F_2\left(\tau;\delta,\beta;z\right), $$

$$ W_{-\frac{1}{2},1}^{\tau,\delta}\left(-z\right)=1-\frac{\tau z}{\delta\sqrt{\pi}}\,{}_3F_3\left(\frac{1}{2},\frac{1+\tau}{2},\frac{2+\tau}{2};\frac{3}{2},\frac{1+\delta}{2},\frac{2+\delta}{2};-\frac{z^2}{4}\right), $$

respectively, the required results are obtained.

3. Generalized beta distribution and generalized Gauss and confluent hypergeometric Function

If a and b satisfy the condition $-\infty < a,b < \infty$, $p,q > 0$ and $\alpha,\beta,\tau,\delta \in \mathbb{C}$, then the generalized beta function is given as follow:

$$ f\left(y\right)=\begin{cases}\dfrac{1}{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a,b\right)}\,y^{y_1-1}\left(1-y\right)^{y_2-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\dfrac{q_1}{y}-\dfrac{q_2}{1-y}\right), & 0<y<1,\\ 0, & \text{otherwise.}\end{cases} $$

For real number σ, then the σth moment of X is as follow:

$$ E\left(X^{\sigma}\right)=\frac{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a+\sigma,b\right)}{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a,b\right)}, \qquad \left(a,b\in R,\alpha,\beta,\tau,\delta\in C\right). $$

If $\sigma=1$, the mean of the distribution as follow:

$$\mu = E\left(X\right) = \frac{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a+1,b\right)}{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a,b\right)},\ \left(a,b \in R, \alpha,\beta,\tau,\delta \in C\right).$$

The variance of the distribution is obtained as follow:

$$Var = \frac{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a,b\right)\ {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a+1,b\right) - \left\{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a+1,b\right)\right\}^2}{\left\{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a,b\right)\right\}^2}.$$

The moment generating function of the distribution is as follow:

$$M\left(t\right) = \frac{1}{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a,b\right)} \sum_{u=0}^{\infty} {}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a+u,b\right) \frac{t^u}{u!}.$$

The cumulative distribution is given below:

$$F\left(x\right) = \frac{{}^{W}B_{q_1,q_2,x}^{(\alpha,\beta;\tau,\delta)}\left(a,b\right)}{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(a,b\right)}.$$

where

$${}^{W}B_{q_1,q_2,x}^{(\alpha,\beta;\tau,\delta)}\left(a,b\right) = \int_0^x y^{y_1-1}\left(1-y\right)^{y_2-1} W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y} - \frac{q_2}{1-y}\right) dy,$$
$$\left(a,b \in R, \alpha,\beta,\tau,\delta \in C\right),$$

is the generalized incomplete beta function.

Generalized beta function in equation (14) is used to generalize the Gauss hypergeometric and confluent hypergeometric functions as follows:

$${}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;z\right) = \sum_{u=0}^{\infty} \left(g_1\right)_u \frac{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2+u,g_3-g_2\right)}{B\left(g_2,g_3-g_2\right)} \frac{z^u}{u!}, \tag{32}$$

$$\left(\alpha,\beta,\tau,\delta \in C, q_1 \geq 0, q_2 \geq 0, \left|z\right| < 1; \mathrm{Re}\left(g_3\right) > \mathrm{Re}\left(g_2\right) > 0\right).$$

and

$$\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2;g_3;z\right) = \sum_{u=0}^{\infty} \frac{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2+u,g_3-g_2\right)}{B\left(g_2,g_3-g_2\right)} \frac{z^u}{u!}, \tag{33}$$

$$\left(\alpha,\beta,\tau,\delta \in C, q_1 \geq 0, q_2 \geq 0; \mathrm{Re}\left(g_3\right) > \mathrm{Re}\left(g_2\right) > 0\right).$$

Theorem 3.1: The following integral formulas of ${}^{W}F_{q_1,q_2}^{(\propto,\beta;\tau,\delta)}(a,b;c;z)$ and ${}^{W}\Phi_{q_1,q_2}^{(\propto,\beta;\tau,\delta)}(b;c;z)$ hold:

$$ {}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(g_1,g_2;g_3;z)=\frac{1}{B(g_2,g_3-g_2)}\int_0^1 y^{g_2-1}(1-y)^{g_3-g_2-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right) $$

$$ \times\sum_{u=0}^{\infty}(g_1)_u\frac{(yz)^u}{u!}dt, \tag{34} $$

$$ \left(\alpha,\beta,\gamma,\delta\in C, q_1>0, q_2>0; q_1=0, q_2=0 \text{ and } |z|<1; \operatorname{Re}(g_3)>\operatorname{Re}(g_2)>0\right), $$

$$ {}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(g_1,g_2;g_3;z)=\frac{1}{B(g_2,g_3-g_2)}\int_0^1 y^{g_2-1}(1-y)^{g_3-g_2-1}(1-yz)^{-g_1} $$

$$ \times W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy, \tag{35} $$

$$ \left(|\arg(1-z)|<\pi; q_1,q_2,\alpha,\beta,\tau,\delta\in R^{+}, \operatorname{Re}(g_3)>\operatorname{Re}(g_2)>0\right). $$

Proof: Equation (34) can be obtained from (32) and (14) as follow

$$ {}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(g_1,g_2;g_3;z)=\frac{1}{B(g_2,g_3-g_2)}\sum_{u=0}^{\infty}\int_0^1 y^{g_2+u-1}(1-y)^{g_3-g_2-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right) $$

$$ \times\sum_{u=0}^{\infty}(g_1)_u\frac{z^u}{u!}dy. $$

On interchanging the order of integration and summation and simplifying, the required result in (34) is obtained.

Equation (35) can be obtained from (34) and the identity $\sum_{u=0}^{\infty}(g_1)_u\frac{(yz)^u}{u!}=(1-zy)^{-g_1}$, $|z|<1$.

Corollary 3.1: The following Kumar confluent hypergeometric function formulas hold:

$$ {}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(g_2;g_3;z)=\frac{1}{B(g_2,g_3-g_2)}\int_0^1 y^{g_1-1}(1-y)^{g_3-g_2-1}e^{zy}\,W_{\alpha,\beta}^{\gamma,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy, $$

$$ {}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(g_2;g_3;z)=\frac{\exp(z)}{B(g_2,g_3-g_2)}\int_0^1 y^{g_3-g_2-1}(1-y)^{g_2-1}\exp(-zy)W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{1-y}-\frac{q_2}{y}\right)dy. $$

$$ \left(|\arg(1-z)|<\pi, q_1,q_2,\alpha,\beta,\tau,\delta\in R^{+}, \operatorname{Re}(g_3)>\operatorname{Re}(g_2)>0\right). $$

Theorem 3.2: The following differentiation formulas of ${}^{W}F_{q_1,q_2}^{(\propto,\beta;\tau,\delta)}(g_1,g_2;g_3;z)$ and ${}^{W}\Phi_{q_1,q_2}^{(\propto,\beta;\tau,\delta)}(g_2;g_3;z)$hold.

$$\frac{d}{dz}\,{}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;z\right)=\frac{g_1g_2}{g_3}\,{}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1+1,g_2+1;g_3+1;z\right), \tag{36}$$

$$\frac{d^u}{dz^u}\,{}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;z\right)=\frac{\left(g_1\right)_u\left(g_2\right)_u}{\left(g_3\right)_u}\,{}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1+u,g_2+u;g_3+u;z\right). \tag{37}$$

Proof: Differentiating equation (32) with respect to z, yield

$$\frac{d}{dz}\,{}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;z\right)=\sum_{u=0}^{\infty}\left(g_1\right)_u\frac{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2+u,g_3-g_2\right)}{B\left(g\ ,g_3-g_2\right)}\frac{z^{u-1}}{(u-1)!}$$

Setting $u\rightarrow u+1$ and the relations

$$\left(g_1\right)_{u+1}=g_1\left(g_1+1\right)_u \text{ and } B\left(g_2,g_3-g_2\right)=\frac{g_2}{g_3}B\left(g_2+1,g_3-g_2\right), \tag{38}$$

gives the desired result in (36). Continue differentiating (36) $(n-1)$-times gives (37).

Corollary 3.2: The following confluent hypergeometric differential formulas hold:

$$\frac{d}{dz}\,{}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2;g_3;z\right)=\frac{g_2}{g_3}\,{}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2+1;g_3+1;z\right), \tag{39}$$

$$\frac{d^u}{dz^u}\,{}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2;g;z\right)=\frac{\left(g_2\right)_u}{\left(g_3\right)_u}\,{}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2+u;g+u;z\right).$$

Theorem 3.3: The following transformations of ${}^{W}F_{q_1,q_2}^{(\propto,\beta;\tau,\delta)}(g_1,g_2;g_3;z)$ and ${}^{W}\Phi_{q_1,q_2}^{(\propto,\beta;\tau,\delta)}(g_2;g_3z)$ holds true.

$${}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;z\right)=\left(1-z\right)^{-g_1}\,{}^{W}F_{q_2,q_1}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_3-g_2;g_3;\frac{z}{z-1}\right). \tag{40}$$

Proof: Equation (40), can be obtained by using $[1-z(1-t)]^{-g_1}=(1-z)^{-g_1}(1-\frac{z}{z-1}t)^{-g_1}$ and replacing t by $1-t$ in (35),

$${}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;z\right)=\frac{1}{B\left(g_2,g_3-g_2\right)}\int_0^1 y^{g_3-g_2-1}\left(1-y\right)^{g_2-1}\left(1-z\right)^{-g_1}\left(1+\frac{z}{1-z}y\right)^{-g_1}$$

$$\times W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{1-y}-\frac{q_2}{y}\right)dy.$$

On simplifying, the desired result in (40) is obtained.

Corollary 3.3: The following generalized confluent hypergeometric transformation formula hold true:

$${}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2;g_3;z\right)=\exp(z)\,{}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_3-g_2;g_3;-z\right).$$

Theorem 3.4: The following differential and difference relations of ${}^{W}F_{q_1,q_2}^{(\propto,\beta;\tau,\delta)}(g_1,g_2;g_3;z)$ and ${}^{W}\Phi_{q_1,q_2}^{(\propto,\beta;\tau,\delta)}(g_2;g_3 z)$ holds true.

$$\Delta_{g_1}\,{}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;z\right)=\frac{g_2 z}{g_3}\,{}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1+1,g_2+1;g_3+1;z\right), \tag{41}$$

$$\frac{d}{dz}\,{}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;z\right)=\frac{g_1}{z}\Delta_{g_1}\,{}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;z\right), \tag{42}$$

$$g_2\,\Delta_{g_2}\,{}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2;g_3+1;z\right)+g_3\,\Delta_{g_3}\,{}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2;g_3;z\right)=0, \tag{43}$$

$$\frac{d}{dz}\,{}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2;g_3;z\right)=\frac{g_2}{g_3}\,{}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2;g_3+1;z\right)-\Delta_{g_3}\,{}^{W}\Phi_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2;g_3;z\right). \tag{44}$$

Proof: Equation (41) can be obtained by using integral representation in (35) of the new generalized Gauss hypergeometric function and Δ_{g_1} is the shifted operator with respect to g_1,

$$\Delta_{g_1}\,{}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;z\right)={}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1+1,g_2;g_3;z\right)-{}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;z\right) \tag{45}$$

$$=\frac{z}{B\left(g_2,g_3-g_2\right)}\int_0^1 y^{g_2}\left(1-y\right)^{g_3-g_2-1}\left(1-zy\right)^{-g_1-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy.$$

Setting $g_1\rightarrow g_1+1$, $g_2\rightarrow g_2+1$ and $g_3\rightarrow g_3+1$, respectively, in equation (45).

$${}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1+1,g_2+1;g_3+1;z\right)=\frac{b}{g_3\,B\left(g_2,g_3-g_2\right)}\int_0^1 y^{g_2}\left(1-y\right)^{g_3-g_2-1}\left(1-zy\right)^{-g_1-1}$$

$$\times W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy. \tag{46}$$

Using the result in equation (14) in (46), the desired result in (41) is obtained.

Equation (42) can be obtained by using differentiation formula in (36) and considering (38), the required result is obtained.

Equation (43) can be obtained by replacing g_2 and g_3 by g_2+1 and g_3+1 in equation (33), respectively and some simplifications, the required result is obtained.

Equation (44) can be obtained by using differentiation formula in (39) and (43).

The following theorems for the new generalized Gauss hypergeometric and confluent hypergeometric functions.

Theorem 3.5: The following summation theorem of ${}^{W}F_{q_1,q_2}^{(\propto,\beta;\gamma\tau\delta)}(g_1,g_2;g_3;z)$ and ${}^{W}\Phi_{q_1,q_2}^{(\propto,\beta;\tau,\delta)}(g_2;g_3;z)$ holds true.

$$ {}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(g_1,g_2;g_3;1)=\frac{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(g_2,g_3-g_1-g_2)}{B(g_2,g_3-g_2)}. \tag{47}$$

Proof: From equation (35), substitute $z=1$, yield

$$ {}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(g_1,g_2;g_3;1)=\frac{1}{B(g_2,g_3-g_2)}\int_0^1 y^{g_2-1}(1-y)^{g_3-g_1-g_2-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy.$$

On simplifying, the required result is obtained.

Remark 3.1: The formula in equation (47) in the case $q_1=q_2=0$, $\alpha=0$, $\beta=2$ and $\tau=\delta$ reduces to the well-known classical Gauss summation formula (see for example, Rao & Lakshminarayanan, 2018).

Theorem 3.6: The following generating function of ${}^{W}F_{q_1,q_2}^{(\propto,\beta;\gamma\tau\delta)}(g_1,g_2;g_3;z)$ and ${}^{W}\Phi_{q_1,q_2}^{(\propto,\beta;\tau,\delta)}(g_2;g_3;z)$ holds true.

$$\sum_{u=0}^{\infty}(g_1)_u\ {}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(g_1+u,g_2;g_3;z)\frac{y^u}{u!}=(1-y)^{-g_1}\ {}^{W}F_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2;g_3;\frac{z}{1-t}\right), \tag{48}$$

$$\left(q_1\geq 0,q_2\geq 0,|y|<1\right).$$

Proof: Setting U in the left-hand sides of equation (48), and simplification,

$$U=\sum_{u=0}^{\infty}(g_1)_u\left\{\sum_{n=0}^{\infty}(g_1+u)_n\frac{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}(g_1+n,g_2-g_3)}{B(g_1,g_2-g_3)}\frac{z^n}{n!}\right\}\frac{y^u}{u!}.$$

Reversing the order of summations and using the identity in equation (38), gives

$$U=\sum_{r=0}^{\infty}(g_1)_r\frac{{}^{W}B_{p,q}^{(\alpha,\beta;\tau,\delta)}(g_1+n,g_2-g_3)}{B(g_1,g_2-g_3)}(1-t)^{-g_1-n}\frac{z^n}{n!},\left\{\sum_{n=0}^{\infty}(g_1+n)_n\frac{t^r}{r!}=(1-t)^{-g_1-n}\right\}.$$

On simplifying, the desired result is obtained.

4. Generalized hypergeometric functions of two and three variables and their integral representations

In this section, generalization of the first two Appell's hypergeometric functions and Lauricella's hypergeometric function of three variables is used (Liu, 2014; Agarwal *et al.*, 2015).

$$ {}^{W}F_{1,q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2,g_3;g_4;t;z\right)=\sum_{u,s=0}^{\infty}\left(g_2\right)_u\left(g_3\right)_s\frac{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1+u+s,g_4-g_1\right)}{B\left(g_1,g_4-g_1\right)}\frac{t^u}{u!}\frac{z^s}{s!}, \quad (49) $$

$$ \left(|t|<1,|z|<1\right). $$

$$ {}^{W}F_{2,q_1,q_2}^{(\alpha,\beta;\tau,\delta,\alpha',\beta';\tau',\delta')}\left(g_1,g_2,g_3;g_4,g_5;t;z\right)=\sum_{u,s=0}^{\infty}\left(g_1\right)_{u+s}\frac{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_2+u,g_4-g_2\right)}{B\left(g_2,g_2-g_2\right)} $$

$$ \times\frac{{}^{W}B_{q_1,q_2}^{(\alpha',\beta';\tau',\delta')}\left(g_3+s,g_5-g_3\right)}{B\left(g_3,g_5-g_3\right)}\frac{t^u}{u!}\frac{z^s}{s!}, \quad (50) $$

$$ \left(|t|+|z|<1\right). $$

$$ {}^{W}F_{D,q_1,q_2}^{(3;\alpha,\beta;\tau,\delta)}\left(g_1,g_2,g_3,g_4;g_5;t;x;z\right)=\sum_{u,s,h=0}^{\infty}\frac{{}^{W}B_{q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1+u+s+h,g_5-g_1\right)}{B\left(g_1,g_5-g_1\right)}\left(g_2\right)_u $$

$$ \times\left(g_3\right)_s\left(g_4\right)_h\frac{t^r}{u!}\frac{x^s}{s!}\frac{z^h}{h!}, \quad (51) $$

$$ \left(|t|<1,|x|<1,|z|<1\right). $$

where ${}^{W}F_{1,q_1,q_2}^{(\propto,\beta;\tau,\delta)}(g_1g_2,,g_3;g_4;x;y)$ and ${}^{W}F_{2,q_1,q_2}^{(\propto,\beta;\tau,\delta,\propto',\beta';\tau',\delta')}(g_1,g_2,g_3,g_4;g_5;x;y;z)$ are the first two Appell's hypergeometric functions of two variables and ${}^{W}F_{D,q_1,q_2}^{(3;\propto,\beta;\tau,\delta)}(g_1,g_2,g_3,g_4;g_5;x;y;z)$ is the Lauricella's hypergeometric function of three variables.

Theorem 4.1: The following integral representations of ${}^{W}F_{1,q_1,q_2}^{(\propto,\beta;\tau,\delta)}(g_1,g_2,g_3;g_4;x;y)$ holds true.

$$ {}^{W}F_{1,q_1,q_2}^{(\alpha,\beta;\tau,\delta)}\left(g_1,g_2,g_3;g_4;t;x\right)=\frac{1}{B\left(g_1,g_4-g_1\right)}\int_0^1\frac{y^{g_1-1}\left(1-y\right)^{g_4-g_1-1}}{\left(1-ty\right)^{g_2}\left(1-xy\right)^{g_3}} $$

$$\times W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy, \tag{52}$$

$$\left(\max\left\{|t|,|x|\right\}<1;\operatorname{Re}(q_1)>0,\operatorname{Re}(q_2)>0;\min\left\{\operatorname{Re}(\alpha),\operatorname{Re}(\beta),\operatorname{Re}(\tau),\operatorname{Re}(\delta)\right\}>0\right).$$

Proof: Let U denote the left-hand sides of (52), then using (49) and simplifications,

$$U=\sum_{\mathrm{u},s=0}^{\infty}\int_0^1 y^{g_1+u+s-1}(1-y)^{g_4-g_1-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)\frac{(g_2)_u(g_3)_s}{B(\mathrm{g}_1,\mathrm{g}_4-g_1)}\frac{t^u}{\mathrm{u}!}\frac{x^s}{s!}dy.$$

Interchanging the order of summation and integration in equation,

$$U=\frac{1}{B(\mathrm{g}_1,\mathrm{g}_4-g_1)}\int_0^1 y^{g_1-1}(1-y)^{g_4-g_1-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)\left\{\sum_{\mathrm{u},s=0}^{\infty}(g_2)_r\frac{(ty)^u}{\mathrm{u}!}\right\}$$

$$\times\left\{\sum_{s=0}^{\infty}(g_3)_s\frac{(xy)^s}{s!}\right\}dy.$$

On simplifying this equation, the required result in (52) is obtained.

Corollary 4.1: For a bounded sequence of essentially arbitrary complex numbers, (Liu 2014; Agarwal *et al.* 2015)

$$\sum_{N=0}^{\infty}f(N)\frac{(t+x)^N}{N!}=\sum_{u=0}^{\infty}\sum_{s=0}^{\infty}f(u+s)\frac{t^u}{\mathrm{u}!}\frac{x^s}{s!}. \tag{53}$$

Theorem 4.2: The following integral representation of ${}^{\mathrm{W}}F_{2,q_1,q_2}^{(\alpha,\beta;\tau,\delta,\alpha\prime,\beta\prime;\tau\prime,\delta\prime)}(g_1,g_2,g_3,g_4;g_5;x;y;z)$ holds true.

$${}^{W}F_{2,\mathrm{q}_1,q_2}^{(\alpha,\beta;\tau,\delta,\alpha',\beta';\tau',\delta')}(\mathrm{g}_1,\mathrm{g}_2,\mathrm{g}_3;\mathrm{g}_4,\mathrm{g}_5;\mathrm{t};\mathrm{x})=\frac{1}{B(\mathrm{g}_2,\mathrm{g}_4-g_2)B(\mathrm{g}_3,\mathrm{g}_5-g_3)}\int_0^1\int_0^1\frac{(1-y)^{g_4-g_2-1}}{(1-yt-xc)^{g_1}}$$

$$\times y^{g_2-1}c^{g_3-1}(1-c)^{g_5-g_3-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)W_{\alpha',\beta'}^{\tau',\delta'}\left(-\frac{q_1}{c}-\frac{q_2}{1-c}\right)dydc, \tag{54}$$

$$\left(|t|+|x|<1\right).$$

Proof: Let U denote the left-hand side of (54). Then using (50),

$$U=\sum_{\mathrm{u},s=0}^{\infty}\left\{\int_0^1 y^{g_2+u-1}(1-y)^{g_4-g_2-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy\right\}\frac{(g_1)_{u+s}}{B(\mathrm{g}_2,\mathrm{g}_4-g_2)B(\mathrm{g}_3,\mathrm{g}_5-g_3)}$$

$$\times\left\{\int_0^1 c^{g_3+s-1}(1-c)^{g_5-g_3-1}W_{\alpha',\beta'}^{\tau',\delta'}\left(-\frac{q_1}{c}-\frac{q_2}{1-c}\right)dc\right\}\frac{x^u}{u!}\frac{y^s}{s!}.$$

Interchanging the order of summation and integration,

$$U=\frac{1}{B(g_2,g_4-g_2)B(g_3,g_5-g_3)}\int_0^1\int_0^1 y^{g_2-1}c^{g_3-1}(1-y)^{g_4-g_2-1}(1-c)^{g_5-g_3-1}$$

$$\times W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_1}{1-y}\right)W_{\alpha',\beta'}^{\tau',\delta'}\left(-\frac{q_1}{c}-\frac{q_2}{1-c}\right)\left(\sum_{u,s=0}^{\infty}(g_1)_{u+s}\frac{(xy)^u}{u!}\frac{(xc)^s}{s!}\right)dydc.$$

Applying equation (53) gives the required result in (54).

Theorem 4.3: The following integral representation of ${}^{W}F_{D,q_1,q_2}^{(3,\alpha,\beta,\tau,\delta)}(g_1,g_2,g_3,g_4;g_5;t;x;z)$ holds true.

$${}^{W}F_{D,q_1,q_2}^{(3;\alpha,\beta;\tau,\delta)}(g_1,g_2,g_3,g_4;g_5;t;x;z)=\frac{1}{B(g_1,g_5-g_1)}\int_0^1\frac{y^{g_1-1}(1-y)^{g_3-g_1-1}}{(1-xy)^{g_2}(1-yx)^{g_3}(1-zy)^{g_4}}$$

$$\times W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)dy,\qquad (|t|<1,|x|<1,|y|<1). \tag{55}$$

Proof: Let U denote the left-hand side of (55). Then by using (51), gives

$$U=\sum_{u,s,h=0}^{\infty}\int_0^1 y^{g_1+r+s+h-1}(1-y)^{g_3-g_1-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)\frac{(g_2)_u(g_3)_s(g_4)_h}{B(g_1,g_5-g_1)}\frac{t^u}{u!}\frac{x^s}{s!}\frac{z^h}{h!}dy.$$

Interchanging the order of summation and integration, gives

$$U=\frac{1}{B(g_1,g_5-g_1)}\int_0^1 y^{g_1-1}(1-y)^{g_5-g_1-1}W_{\alpha,\beta}^{\tau,\delta}\left(-\frac{q_1}{y}-\frac{q_2}{1-y}\right)\left\{\sum_{u=0}^{\infty}(g_2)_u\frac{(ty)^u}{u!}\right\}$$

$$\times\left\{\sum_{r=0}^{\infty}(g_3)_s\frac{(xy)^s}{s!}\right\}\left\{\sum_{h=0}^{\infty}(g_4)_h\frac{(zy)^h}{h!}\right\}dy.$$

On simplifying, the required result in (55) is obtained.

5. Conclusion

Generalized Wright function was used to define a new generalized beta function with some of its properties like summation formulas, Integral representations, connections with other special functions such as incomplete gamma, classical beta, classical Wright, hypergeometric, error, Fox-H, Fox-Wright, Meijer-G functions are obtained; Beta distribution together with-it corresponding moment, mean, variance, moment generating function and cumulative distribution are also presented. Moreover, the generalized beta function is used to generalized classical and other related extended Gauss, Kumar confluent, Appell's and Lauricella's hypergeometric functions with their integral representations, differential, difference, summation, and transformations formulas. In its special cases (the new generalized beta function), this generalization includes the extension of beta function which were presented in (Chaudhry & Zubair, 2002; Srivastava & Choi, 2012; Chaudhry 1997; Choi *et al.*, 2014; Ata 2018; Kulip *et al.*, 2020). Some properties of these generalized beta are investigated, Kumar confluent, Appell's and Lauricella's hypergeometric functions, most of which are analogous with the classical and other related generalized beta functions. These generalized functions can be used to study theory of fractional integral and differential calculus (see for example, Agarwal *et al.*, 2015; Pucheta, 2017; Rahman *et al.*, 2018; Shadab *et al.*, 2018; Nisar *et al.*, 2019; Singhal and Mittal, 2020) and in the provision of the extended special function such as Mittag-Leffler, Bessel-Maitland and Wright (refer to, Khan *et al.*, 2020a, b).

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