THE STABILITY OF THE CRITICAL POINTS OF THE GENERALIZED GAUSE TYPE PREDATOR-PREY FISHERY MODELS WITH PROPORTIONAL HARVESTING AND TIME DELAY

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ABSTRACT

In the marine ecosystem, the time delay or lag may occur in the predator response function, which measures the rate of capture of prey by a predator. This is because, when the growth of the prey population is null at the time delay period, the predator's growth is affected by its population and prey population densities only after the time delay period. Therefore, the generalized Gause type predator-prey fishery models with a selective proportional harvesting rate of fish and time lag in the Holling type II predator response function are proposed to simulate and solve the population dynamical problem. From the mathematical analysis of the models, a certain dimension of time delays in the predator response or reaction function can change originally stable non-trivial critical points to unstable ones. This is due to the existence of the Hopf bifurcation that measures the critical values of the time lag, which will affect the stabilities of the non-trivial critical points of the models. Therefore, the effects of increasing and decreasing the values of selective proportional harvesting rate terms of prey and predator on the stabilities of the non-trivial critical points of the fishery models were analysed. Results have shown that, by increasing the values of the total proportion of prey and predator harvesting denoted by $q_x E_x$ and $q_y E_y$ respectively, within the range $0.3102 \leq q_x E_x \leq 0.9984$ and $0.5049 \leq q_y E_y \leq 0.5363$, the originally unstable non-trivial critical points of the fishery models can be stable.

1.   Introduction

All living organisms like plants, animals and microorganisms and nonliving organisms like air, soil and water have complex relationships between them in the ecosystem. The relationship between prey and its predator both fish species is known as the predator-prey relationship. Predators are dependent on prey for their continued survival. To avoid a single species from being dominant, the predator is the leading role in maintaining the prey population and improving the biodiversity within the ecosystem. Mathematical models can be applied to simulate the predator-prey population dynamics of the fish species. One of the models to analyse the predator-prey interaction is the generalized Gause type predator-prey model which

is based on the Logistic growth model and predator response function. A study by Hwang (1999) used a unique limit cycle for Gause type predator-prey models for systems of nonlinear ordinary differential equations since the limit cycle does not exist in the linear ordinary systems. According to Feller (1940), almost all populations that increase asymptotically will follow the Logistic growth law to some degree. For the predator-prey relationship, the response functions between prey and predator act as the main indicator of the survival of both populations in an ecosystem. Three functional responses of the predator are Holling type I, Holling type II and Holling type III (Holling, 1965).

The Holling type I predator reaction function is a linear functional response that gives a directly proportional relationship between the rate of consumption of an individual predator and the density of its prey. Besides, the Holling type II predator reaction function is also called a hyperbolic or Michaelis-Menten functional response is used to analyse the behaviour of the predator population towards its different prey densities. The Holling type III or sigmoidal-shaped functional response was originally proposed as a model to describe the transformation of the predator's preference from primary prey to alternative prey when the primary prey is at low densities (Kar & Matsuda, 2007). Safaa *et al.* (2016) analysed that the Holling type IV predator reaction function, also known as the Monod-Haldane functional response, is used to describe a condition in which the predator's per capita rate of predation decreases at sufficiently high prey densities.

Jha & Ghorai (2017) have considered predator-prey models with Holling type I and II predation response functions with selective constant harvesting in the predator-prey systems. Meanwhile, Martin & Ruan (2001) analysed the effects of constant-rate harvesting of prey populations and time lag on the dynamics of a predator-prey system in predation response function. Sinha & Chanda (2014) revealed that the time delay is considered in the predator-prey system because both predator and prey need their own time to grow from birth to maturity and from maturity to death. Moghadas *et al.* (2004) studied a non-standard numerical scheme for a generalized Gause type predator-prey model. They used the Holling type II predator reaction function $\phi(x)$, denoted by:

$$\phi(x) = \frac{mx}{a + x} \tag{1}$$

where $m > 0$ is the capturing rate of prey by a predator, a is the half saturation constant for predator population, which is the prey density at which the functional response of predator is half maximal and x is the number of the prey population. Most researchers (Jha & Ghorai, 2017; Martin & Ruan, 2001) used Holling type II predator response function with selective constant harvesting. However, this research applies selective proportional harvesting. A study by Hntsa & Mengesha (2016) indicated that to prevent the extinction of renewable resources, researchers must devise efficient harvesting strategies that produce optimum yield while maintaining renewable resources above a sustainable level. For fishery management and many other harvesting situations, it is unrealistic to consider that the harvest rate is constant which does not depend on the population, because the more fish in an area, the more will be caught.

This article begins with the determination of the first critical values of the time delay and is followed by the estimation of the critical values of the total proportion of prey and predator harvesting that can affect the stabilities of non-trivial critical points for two fishery models.

2.   Non-Trivial Critical Points and Characteristic Equations of Fishery Models

In this section, the derivation of the non-trivial critical points which are corresponding to both prey and predator presence in an ecosystem and characteristic equations of fishery models I and II are considered. The critical points can also be called equilibrium or fixed points, which can be analysed as steady-state solutions of a dynamical system. The steady-state solutions have the properties of stabilities. According to Zill & Cullen (2009) from the characteristic equation, the type of stability for the critical point can be determined.

2.1 Non-Trivial Critical Point and Characteristic Equation of Fishery Model I

The dynamical system for fishery model I is also known as the generalized Gause type predator-prey fishery model with prey proportional harvesting and the time lag in the Holling type II predator reaction function is presented by:

$$\left.\begin{array}{c} dx(t)/dt = rx(t)\left(1 - x(t)/k\right) - \left(mx(t)y(t)/(a + x(t))\right) - q_x E_x x(t) \\[4mm] dy(t)/dt = -wy(t) + \left(cmx(t - \tau)y(t)/(a + x(t - \tau))\right) \end{array}\right\} \tag{2}$$

where,

x is the number of preys in tonnes
y is the number of predators in tonnes
r is the growth rate of prey population
k is carrying capacity of the prey population
m is the capturing rate of prey by a predator
a is half saturation constant for the predator population
q_x is the catchability coefficient for prey population
E_x is the effort harvesting rate of prey population
w is a death rate of the predator population
τ is the time delay in months
t is time in months
c is the rate of conversion of consumed prey to predator

To derive the non-trivial critical point (x^*, y^*), $dx(t)/dt$ and $dy(t)/dt$ in Equation (2) must be zero. Therefore, the formula for x^* and y^* are:

$$x^* = aw/(cm - w) \tag{3}$$

$$y^* = \left(rx^*(1 - x^*/k) - q_x E_x x^*\right)/\left(mx^*/(a + x^*)\right) \tag{4}$$

By substituting the value of parameters from Kar (2003) which are:
$r = 2, k = 100, a = 20.03, q_x E_x = 0.224, w = 0.5, c = 0.7, m = 1, x(0) = 100$ and $y(0) = 25$, the critical or fixed point obtained is (50.0750, 54.2963).

According to Martin & Ruan (2001) and Zill & Cullen (2009), $\overline{X} = x - x^*$ and $\overline{Y} = y - y^*$. Then the linearized system becomes:

$$\left.\begin{array}{c} d\overline{X}/dt = \left(\left((a + x^*)^2(kr - 2rx^* - kq_x E_x) - akmy^*\right)\overline{X}(t)\right)/\left(k(a + x^*)^2\right) - \left(mx^*\overline{Y}(t)/(a + x^*)\right) \\[4mm] d\overline{Y}/dt = \left(cmy^* a\overline{X}(t - \tau)\right)/(a + x^*)^2 \end{array}\right\} \tag{5}$$

From Equation (5), the characteristic equation (Abdallah *et al.,* 2011; Martin & Ruan, 2001; Ruan, 2009; Xia *et al.*, 2009) is:

$$\lambda^2 + p_x \lambda + r_x e^{-\lambda \tau} = 0 \tag{6}$$

where,

$$p_x = \left(-(a+x^*)^2(kr-2rx^*-kq_x E_x)+akmy^*\right)/\left(k(a+x^*)^2\right) \text{ and } r_x = (cm^2 x^* y^* a)/(a+x^*)^3.$$ By

substituting all the parameter values from Kar (2003), the resulted value p_x is 0.4483 and r_x is 0.1106.

2.2 Non-Trivial Critical Point and Characteristic Equation of Fishery Model II

The dynamical system for fishery model II is also known as the generalized Gause type predator-prey fishery model with predator proportional harvesting and the time lag in Holling type II predator reaction function is presented by:

$$\left. \begin{array}{l} dx(t)/dt = rx(t)\left(1-x(t)/k\right)-\left(mx(t)y(t)/(a+x(t))\right) \\[10pt] dy(t)/dt = -wy(t)+\left(cmx(t-\tau)y(t)/(a+x(t-\tau))\right)-q_y E_y y(t) \end{array} \right\} \tag{7}$$

where q_y is the catchability coefficient of the predator and E_y is the effort harvesting rate of the predator. The non-trivial critical point (x^*, y^*) for fishery model II is calculated by letting both $dx(t)/dt$ and $dy(t)/dt$ in Equation (7) with zero and resulted in the coordinates of the critical values:

$$x^* = (aw + aq_y E_y)/(cm - w - q_y E_y) \tag{8}$$

$$y^* = \left(kr(a+x^*) - rx^*(a+x^*)\right)/\left(km\right) \tag{9}$$

By substituting the value of parameters $r = 1.8$, $k = 100$, $a = 10$, $q_y E_y = 0.5$, $w = 0.5, c = 0.6, m = 1.9$, $x(0) = 25$, $y(0) = 25$ from Kar (2003) and Kar & Matsuda (2006), the non-trivial critical point obtained is (71.4286, 22.0408). All parameter values are taken from Kar (2003) and Kar & Matsuda (2006) because these values fit the fishery models and are in the permissible range.

Again by letting $\bar{X} = x - x^*$ and $\bar{Y} = y - y^*$ (Martin & Ruan (2001); Zill & Cullen (2009)), the linearized system becomes:

$$\left. \begin{array}{l} d\bar{X}/dt = \left(\left((a+x^*)^2(kr-2rx^*)-akmy^*\right)/\left(k(a+x^*)^2\right)\right)\bar{X}(t) - mx^*\bar{Y}(t)/(a+x^*) \\[12pt] d\bar{Y}/dt = cmy^* a\bar{X}(t-\tau)/(a+x^*)^2 \end{array} \right\} \tag{10}$$

From Equation (10), the characteristic equation (Abdallah *et al.,* 2011; Martin & Ruan, 2001; Ruan, 2009; Xia *et al.*, 2009) is:

$$\lambda^2 + p_y \lambda + r_y e^{-\lambda\tau} = 0 \tag{11}$$

where $\quad p_y = \left(-(a + x^*)^2(kr - 2rx^*) + akmy^*\right)/\left(k(a + x^*)^2\right) \quad$ and $\quad r_y = cm^2 x^* y^* a/(a + x^*)^3$.
Substituting the parameter values resulted to $p_y = 0.8346$ and $r_y = 0.0632$.

3.   Fishery Model I with Time Delay

For $\tau \neq 0$, $\lambda = i\omega$ is the root of the characteristic equation of Equation (6). Letting $\lambda = i\omega$ into Equation (6) produces:

$$-\omega^2 + p_x i\omega + r_x e^{-i\omega\tau} = 0 \tag{12}$$

Applying Euler's formula, $e^{i(-\omega\tau)} = \cos(-\omega\tau) + i\sin(-\omega\tau)$ the real and imaginary parts of the complex number in Equation (12) are:

$$\left.\begin{array}{lll} \text{Real part} & : & -\omega^2 + r_x \cos(\omega\tau) = 0 \\ \text{Imaginary part} & : & p_x\omega - r_x \sin(\omega\tau) = 0 \end{array}\right\} \tag{13}$$

Squaring and adding both sides of the real and imaginary equations in Equation (13) resulted to:

$$\omega_{\pm}^2 = \left(-p_x^2 \pm \left(p_x^4 + 4r_x^2\right)^{\frac{1}{2}}\right)/2 \tag{14}$$

$\omega_+^2 = 0.0489$ is obtained from the data (Kar, 2003) resulted to $\lambda = \pm 0.2211i$. The value τ_0^+ is found by solving Equation (13) for τ. So, by multiplying $\sin(\omega\tau)$ and $\cos(\omega\tau)$ into real and imaginary parts of Equation (13) respectively, τ_M^+ is:

$$\tau_M^+ = \left(1/\omega_+\right)\tan^{-1}\left(p_x/\omega_+\right) + \left(2M\pi/\omega_+\right), M = 0, 1, 2, \ldots \tag{15}$$

Considering only the first cycle of the arctangent and substituting all parameter values resulted to $\tau_0^+ = 5.0321$. All the calculated values satisfy the following **Lemma 3.1**.

Lemma 3.1: (Martin & Ruan, 2001)
If $p_x > 0$, $r_x > 0$ and $\tau = \tau_0^+$, then the characteristic equation for fishery model I in Equation (6) has a pair of purely imaginary roots, $\pm i\omega_+$.

Then, to see if a Hopf bifurcation occurs or not, the transversality condition below needs to be verified

$$d \, Re \, \lambda_0^+\left(\tau_0^+\right)/d\tau \ > 0 \tag{16}$$

Taking the implicit differentiation on Equation (6) with respect to τ produces

$$(d\lambda/d\tau)^{-1} = \left((2\lambda + p_x)e^{\lambda\tau}\right)/r_x\lambda - (\tau/\lambda) \tag{17}$$

According to Martin & Ruan (2001), the formula to prove the transversality condition is

$$sign\{d(Re\lambda)/d\tau\}_{\lambda=i\omega} = sign\{Re(d\lambda/d\tau)^{-1}\}_{\lambda=i\omega} \qquad (18)$$

where Re is the real part of the complex number and *sign* is the sign of the real number whether positive or negative. By substituting Equation (17) into Equation (18) we obtain

$$sign\{d(Re\lambda)/d\tau\}_{\lambda=i\omega} = sign\{(2\omega^2 + p_x^2)/r_x^2\} \qquad (19)$$

Substituting all parameter values in Equation (19), we get

$$sign\{d(Re\lambda)/d\tau\}_{\lambda=i\omega} = sign\{24.4248\} \qquad (20)$$

Since the value obtained is positive in Equation (20), **Theorem 3.1** below is satisfied.

Theorem 3.1: (Martin & Ruan, 2001)
$p_x > 0$, $r_x > 0$ and $\tau = \tau_0^+$, then the critical point (x^*, y^*) of predator-prey fishery model I has a pair of purely imaginary roots $\pm i\omega^+$ for $\tau = \tau_0^+$. The critical point of fishery model I is stable for $\tau < \tau_0^+$ and unstable for $\tau > \tau_0^+$. The system undergoes a Hopf bifurcation at τ_0^+.

4. Fishery Model II with Time Delay

For $\tau \neq 0$, $\lambda = i\omega$ is the root of Equation (11). The process to determine the eigenvalue, $\lambda = i\omega$ for fishery model II is the same as in fishery model I. Thus, Equation (14) can be written as Equation (21) in the fishery model II.

$$\omega_{\pm}^2 = \left(-p_y^2 \pm \left(p_y^4 + 4r_y^2 \right)^{\frac{1}{2}} \right)/2 \qquad (21)$$

Substituting parameter values from Kar (2003), we obtained $\lambda = \pm 0.0755i$. Meanwhile $\tau_0^+ = 19.6103$ from Equation (15), by considering the value of $p_y = 0.8346$. Both **Lemma 3.1** and **Theorem 3.1** hold for fishery model II.

5. Numerical Simulation

Numerical methods such as the Runge-Kutta first order (RK1), Runge-Kutta third order (RK3), Runge-Kutta fourth order (RK4) and Runge-Kutta-Fehlberg Fourth-Fifth order (RKF 45) have been used to solve the system of differential equations. Mullen (2015) applied and compared numerical methods RK1 and RK4 to solve the Optimal Velocity Model for one- and two-car systems. Meanwhile, the RK3 method was implemented by Rini & Alyauma (2018) to solve the Nth-Order Fuzzy Differential Equations based on the Combination of Arithmetic, Harmonics and Geometrics Means with initial values. Furthermore, Paul *et al.* (2016) applied the RKF 45 method and the Laplace Adomian Decomposition method (LADM) to solve the Lotka Volterra prey-predator model. The results showed that RKF 45 was a suitable and accurate tool for solving the mentioned model. Hence, this research applied RKF 45 based on

Mathews & Fink (2004) on the visualization of the stabilities of the non-trivial critical points of the two fishery models. By using this numerical method, the graphs of the predator and prey versus time and the phase portrait of prey and predator population concerning time can be plotted.

5.1 Effect of Time Delay on the Stability of the Non-Trivial Critical Point of Fishery Model I

In this subsection, we verify all mathematical results $\lambda = \pm 0.2211\,i$ and $\tau = 5.0321$ obtained in section 3 on the stability of the non-trivial critical point of fishery model I at time delays equal to 4.00, 5.0321 and 6.00.
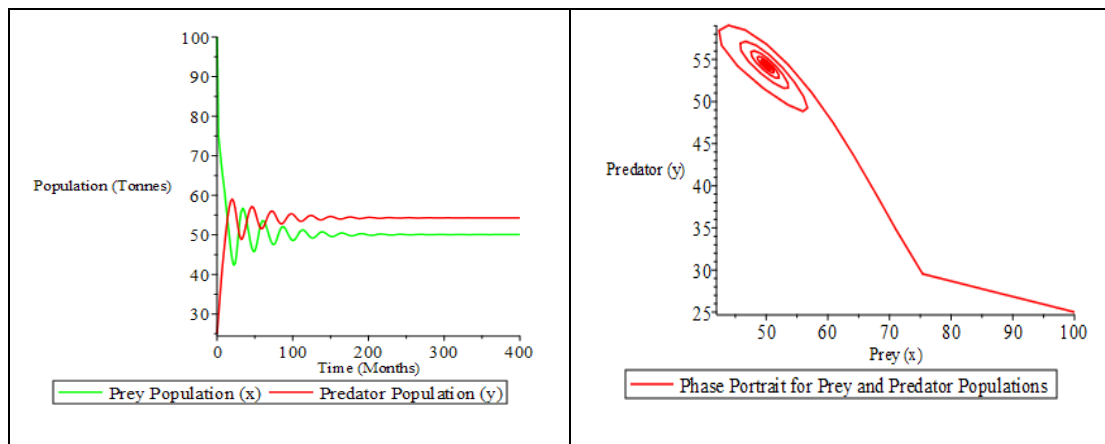

Figure 1(a). Predator and prey population versus time for fishery model I at $\tau = 4.00$.

Figure 1(b). Phase portrait of fishery model I at $\tau = 4.00$.

Based on figure 1(a), periodic oscillations occur on both prey and predator populations at 20 to 175 months. After the 175th month, both prey and predator populations converge to their equilibrium population, which are 50.0750 and 54.2963 tonnes of fishes respectively. In figure 1(b) both prey and predator populations spiral toward their critical values as time increases. At $\tau = 4.00$ the non-trivial equilibrium point $(50.0750, 54.2963)$ will be an asymptotically stable focus.
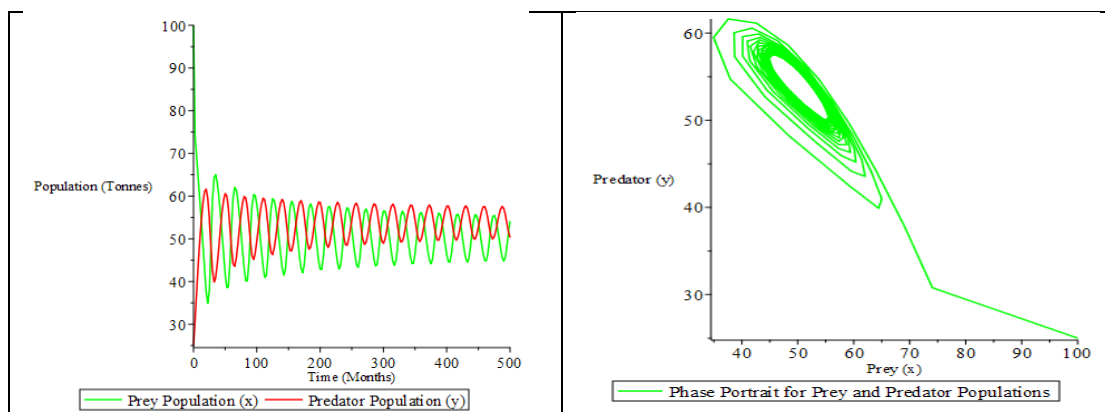

Figure 2(a). Predator and prey population versus time for fishery model I at $\tau = 5.0321$.

Figure 2(b). Phase portrait of fishery model I at $\tau = 5.0321$.

In figures 2(a) and 2(b), both prey and predator populations oscillate about their critical point or in other words, the prey and predator population neither approach nor recede away from the non-trivial critical point (50.0750, 54.2963). The eigenvalues obtained from the characteristic equation in Equation (6) at $\tau = 5.0321$ are $\lambda = \pm 0.2211\,i$. The Hopf bifurcation, which is the visual aspect of the periodic orbit through a local change in the stability properties of a critical point as shown in figures 2(a) and 2(b). The time delay, $\tau = 5.0321$ in fishery model I, indicates the transition from the steady condition of prey and predator population in the marine ecosystem to periodic conditions. Research conducted by Baisad & Moonchai (2018), found that a Hopf bifurcation occurs when a system of differential equations has a pair of complex conjugate eigenvalues of the Jacobian matrix at the fixed point.
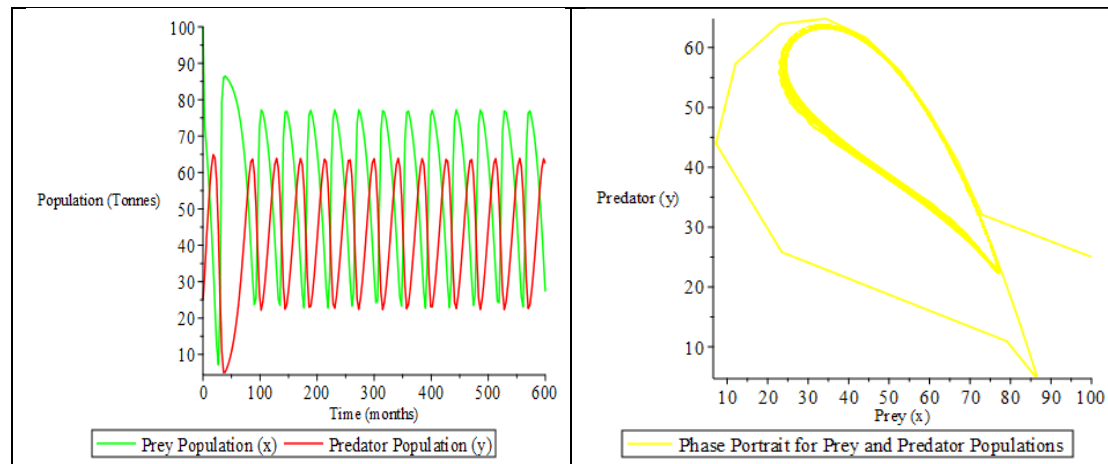

Figure 3(a). Predator and prey population versus time for fishery model I at $\tau = 6.00$.

Figure 3(b). Phase portrait of fishery model I at $\tau = 6.00$.

From the numerical simulation, the limit cycle, which is the periodic solution or orbit, occurs at $\tau = 6.00$, which means that the bifurcation occurs before $\tau = 6.00$. From figures 3(a) and 3(b), there is a bifurcating periodic solution, which causes the non-trivial critical point (50.0750, 54.2963) of the fishery model I to become unstable because both prey and predator populations do not converge to their critical number in finite time (600 months). Figure 3(a) shows the large periodic oscillations on both prey and predator populations around the critical point (50.0750, 54.2963). This indicates that the mentioned critical point is unstable. At this level of time delay, the growth of the predator population is affected by its population and prey population densities that cause the critical point of the model to be unstable.

5.2   Effect of Time Delay on the Stability of the Non-Trivial Critical Point of Fishery Model II

In this subsection, we verify all mathematical results $\lambda = \pm 0.0755\,i$ and $\tau = 19.6103$ obtained in section 4 regarding the stability of the non-trivial fixed point for fishery model II at time delays equal to 15.00, 19.6103 and 21.00.
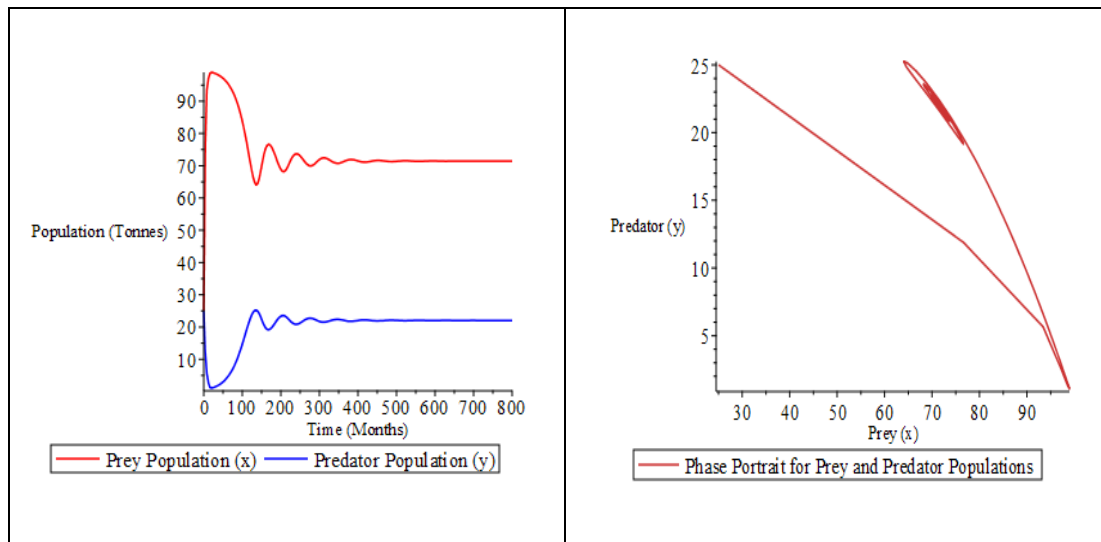
Figure 4(a). Predator and prey population versus time for fishery model II at $\tau = 15.00$.

Figure 4(b). Phase portrait of fishery model II at $\tau = 15.00$.

Figure 4(b) shows that the fixed point is an asymptotically stable focus because both prey and predator populations approach the fixed point (71.4286, 22.0408). In figure 4(a), there is an occurrence of small periodic oscillation for both prey and predator populations from 125 to 375 months. After the 375th month, both prey and predator populations converge to the equilibrium number of populations, 71.4286 and 22.0408 tonnes of fishes respectively.
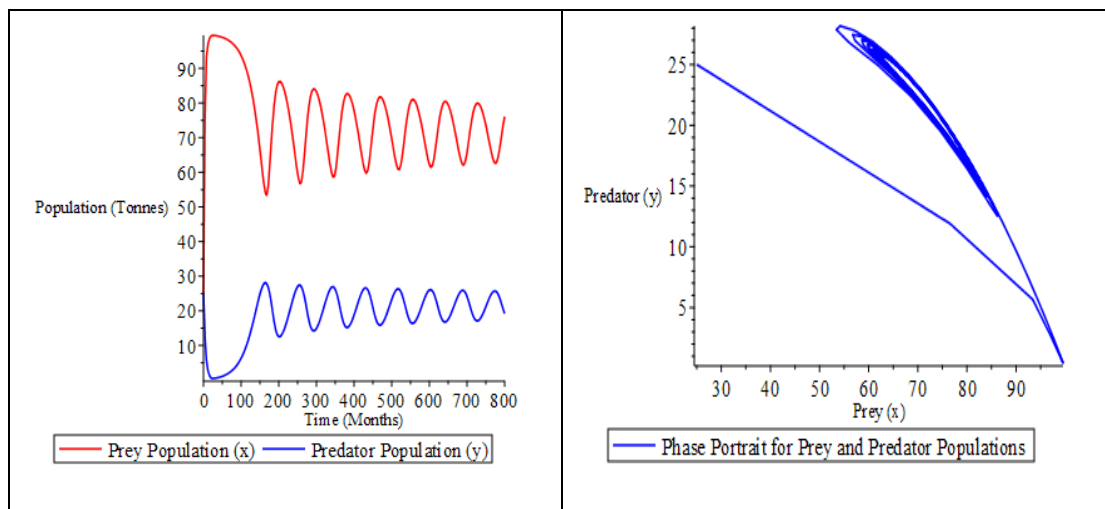

Figure 5(a). Predator and prey population versus time for fishery model II at $\tau = 19.6103$.

Figure 5(b). Phase portrait of fishery model II at $\tau = 19.6103$.

In figure 5(a), both prey and predator populations reach the periodic oscillations that are induced by Hopf bifurcation around the non-trivial critical point up to 800 months. Figure 5(b) shows that both prey and predator populations oscillate about the equilibrium point (71.4286, 22.0408). Prey and predator population neither approach nor recede away from the non-trivial critical point (71.4286, 22.0408). At $\tau = 19.6103$, the roots from the characteristic equation in Equation (11) are $\lambda = \pm 0.0755\,i$. Based on the eigenvalue analysis of the nonlinear dynamical system, when the eigenvalues, λ are pure imaginary numbers, the type of stability of the critical point can either be the stable centre, unstable spiral point or asymptotically stable spiral point in the nonlinear system (Zill & Cullen, 2009).
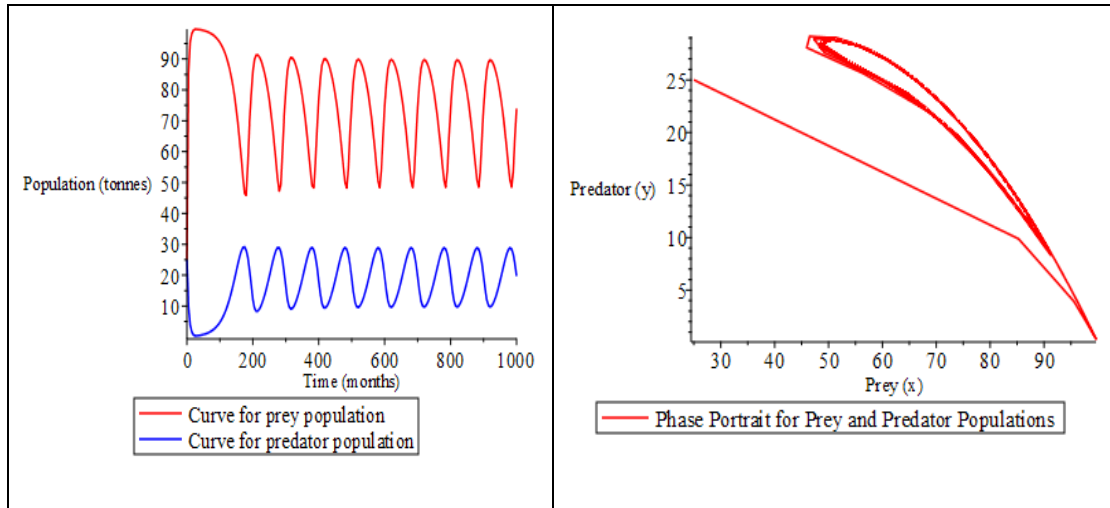
Figure 6(a). Predator and prey population versus time for fishery model II at $\tau = 21.00$.

Figure 6(b). Phase portrait of fishery model II at $\tau = 21.00$.

Figure 6(a) shows the bifurcating periodic solution produces a non-stable equilibrium point at point (71.4286, 22.0408). Thus, both prey and predator populations do not converge at the critical point (71.4286, 22.0408). Figures 6(a) and 6(b) show the limit cycle that describes the occurrence of oscillatory behaviour by the Hopf bifurcation on both prey and predator populations, hence the non-stable critical point does not reach its equilibrium number of populations. At this level of time delay which is $\tau = 21.00$, the growth of predator and prey population is unstable.

5.3 Effect of Total Proportion of Prey Harvesting ($q_x E_x$) on the Stability of the Non-Trivial Critical Point of Fishery Model I

In this subsection, we analyse the critical values of the total proportion of prey harvesting on the fishery model I when $\tau = 6.00$, which can stabilise the non-trivial critical point that is unstable as shown in figure 3(a). From numerical testing, the values $q_x E_x$ within the range [0.3102, 0.9984], changes an originally unstable non-trivial critical point to become stable.
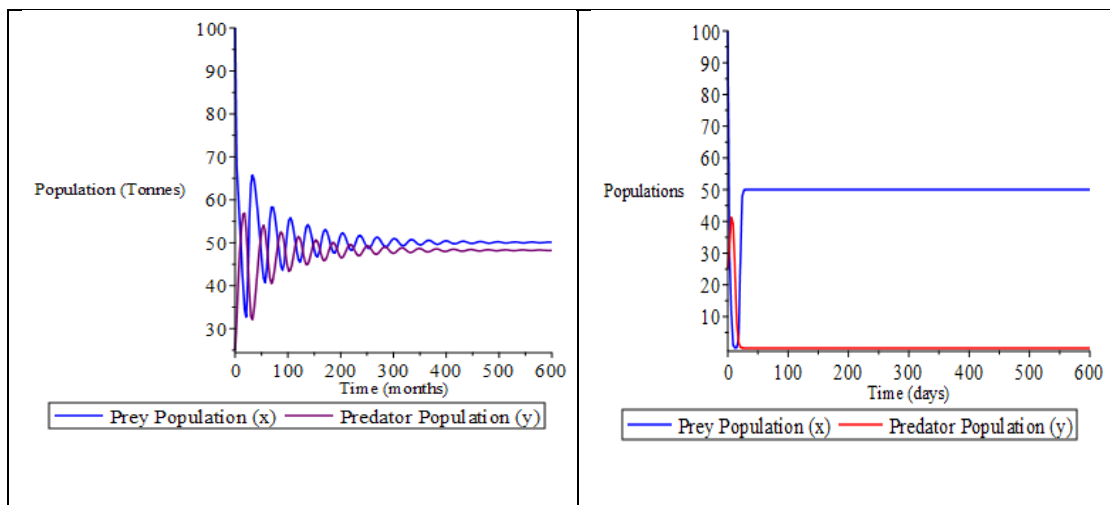

Figure 7. Predator and prey population versus time for fishery model I with $q_x E_x = 0.3102$.

Figure 8. Predator and prey population versus time for fishery model I with $q_x E_x = 0.9984$.

At $q_x E_x = 0.3102$, the originally unstable non-trivial critical point become stable as shown in figure 7. This is because the equilibrium number of predator population y^* from Equation (4) decreases while the equilibrium number of prey population x^* remains constant ($x^* = 50.0750$). There is an oscillation or bifurcation that occurs on both prey and predator populations before the 500th month. As time increases, both prey and predator populations converge to the critical point $(x^*, y^*) = (50.0750, 48.2533)$. From figure 8, $q_x E_x = 0.9984$, both prey and predator populations converge to the critical point $(50.0750, 0.0072)$. The prey population is dominant in the marine ecosystem, while the predator population is driven to extinction. According to Dawed *et al.* (2014), this situation is preferable because the prey population does not receive any threat from the predator population to survive in the marine ecosystem. Thus, the prey population would grow at a natural rate. The competition that is only present is in between the prey population itself, which is hunting for the same food when the predator population is absent in the marine ecosystem. Beyond the critical proportional harvesting rate of prey $(q_x E_x = 0.9985)$, the number of prey populations will be decreased.

5.4   Effect of Total Proportion of Predator Harvesting ($q_y E_y$) on the Stability of the Non-Trivial Critical Point of Fishery Model II

In this subsection, we analyse the critical values of the total proportion of predator harvesting on the fishery model II with $\tau = 21.00$ which can stabilise the non-trivial fixed point that is unstable as shown in figure 6(a). From the numerical simulation, the range value of the parameter $q_y E_y$ that can stabilise the unstable non-trivial critical point is [0.5049, 0.5363].
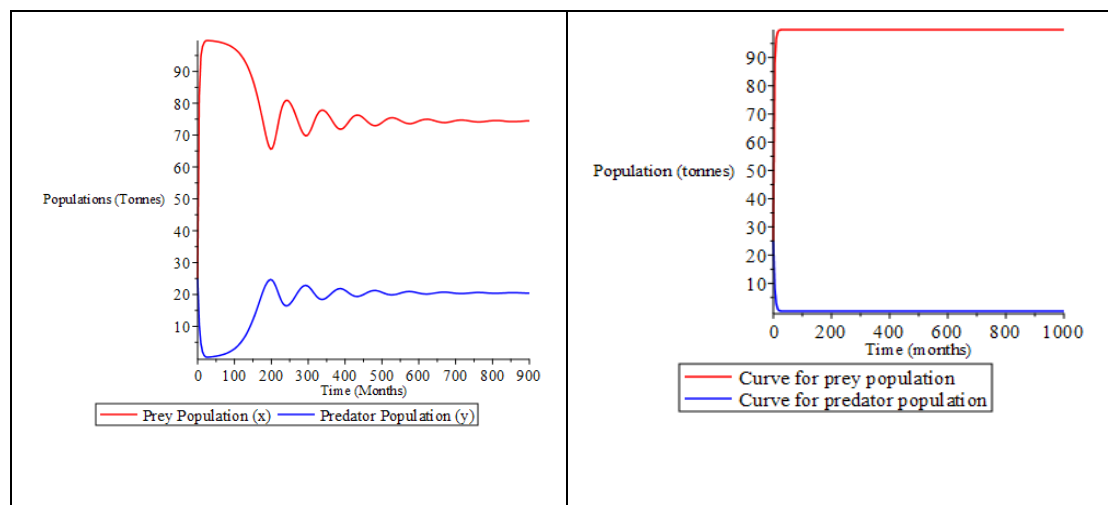

Figure 9. Predator and prey population versus time for fishery model II with $q_y E_y = 0.5049$.

Figure 10. Predator and prey population versus time for fishery model II with $q_y E_y = 0.5363$.

From figure 9, the non-trivial critical point $(x^*, y^*) = (74.3819, 20.4793)$ of fishery model II for $\tau = 21.00$ and $q_y E_y = 0.5049$ is stable, which is described by the small periodic oscillations between 200 to 700 months. After the 700th month, both prey and predator populations will converge to their fixed point $(x^*, y^*) = (74.3819, 20.4793)$ which is the critical point. For $q_y E_y = 0.5363$ and $\tau = 21.00$, both prey and predator populations approach the equilibrium point $(99.9325, 0.0703)$ and stabilise at the beginning of the 25th month, as shown in figure 10. The equilibrium number of predator population approaches null, while the prey population reaches its full capacity. In figures 9 and 10 populations of prey are increasing as the total proportion of predator harvesting is increasing. This is because the more predator is proportionally harvested, the more prey population will be in the marine ecosystem. The prey population does not get any threat anymore from predator populations to survive in the marine ecosystem.

Hence, the predator population is driven to extinction, whereas the prey population is dominant in the marine ecosystem.

6.   Conclusion

From this research, we observed that the first critical values of time lag in the Holling type II predator reaction function, $\tau = 5.0321$ and $\tau = 19.6103$ in fishery model I and II respectively, enable both stable non-trivial critical points to become unstable due to the existence of Hopf bifurcation that may lead to the existence of a limit cycle. We conclude that the proportional harvesting rate of the prey and predator populations had a stabilising effect on the stabilities of the non-trivial equilibrium points of the fishery systems. Applying the parameter values from Kar (2003) and Kar & Matsuda (2006), we obtain $\tau = 5.0321$ and $\tau = 19.6103$, the first critical values of time delay for the fishery models I and II, respectively. We derived new critical or equilibrium points (x^*, y^*) as written in Equation (3),(4),(8) and (9) for both models. The non-trivial critical points became stable at $\tau < 5.0321$ and $\tau < 19.6103$, in fishery models I and II respectively. To stabilise the non-trivial critical points of both fishery models, the total proportion of prey harvesting $q_x E_x$ is in the interval [0.3102, 0.9984] and the total proportion of predator harvesting $q_y E_y$ is given by $0.5049 \leq q_y E_y \leq 0.5363$. We conclude that the non-trivial fixed points become stable when the predator population is reduced in the marine ecosystem using both fishery models I and II, inevitably enhancing the growth of the prey population.

References

Abdallah, C. T., Sipahi, R., Niculescu, S. J., Michiels, W., & Gu, K. (2011). Stability and Stabilization of Systems with Time Delay: Limitations and Opportunities. *IEEE Transactions on Control Systems Technology, 31*(1), 38-65.

Baisad, K., & Moonchai, S. (2018). Analysis of Stability and Hopf Bifurcation in a Fractional Gauss-Type Predator-Prey Model with Allee Effect and Holling Type-III Functional Response. *Advances in Difference Equations, 82*, 1-20. DOI:10.1186/s13662-018-1535-9.

Dawed, M. Y., Koya, P. R., & Goshu, A. T. (2014). Mathematical Modelling of Population Growth: The Case of Logistic and Von Bertalanffy Models. *Open Journal of Modelling and Simulation*, *2*(04),113.

Feller, W. (1940). On Logistic Law of Growth and its Empirical Verification in Biology. *Acta Biotheoretica, 5*(2), 51-66. DOI:10.1007/BF01602862

Hntsa, K. H., & Mengesha, Z. T. (2016). Mathematical Modelling of Fish Resources Harvesting with Predator at Maximum Sustainable Yield. *International Journal of Integrative Sciences, Innovation and Technology (IJIIT), 5*(4),1-11.

Holling, C. S. (1965). The Functional Response of Predators to Prey Density and its Role in Mimicry and Population Regulation. *Memoirs of the Entomological Society of Canada*, *97*,5-60. DOI:10.4039/entm9745fv

Hwang, T.W. (1999). Uniqueness of the Limit Cycle for Gause- Type Predator-Prey Systems. *Journal of Mathematical Analysis and Applications*, 238, 179-195.

Jha, P. K., & Ghorai, S. (2017). Stability of Prey-Predator Model with Holling Type Response Function and Selective Harvesting. *Journal of Applied & Computational Mathematics, 6*(3), 1-7. DOI:10.4172/2168-9679.1000358

Kar, T. K. (2003). Selective Harvesting in a Prey-Predator Fishery with Time Delay. *Mathematical and Computer Modelling*,*38*,449-458. DOI:10.1016/S0895-7177(03)00232-2

Kar, T. K., & Matsuda, H. (2006). Controllability of a Harvested Prey-Predator System with Time Delay. *Journal of Biological Systems, 14*(02), 243-254.

Kar, T. K., & Matsuda, H. (2007). Global Dynamics and Controllability of a Harvested Prey-Predator System with Holling Type III Functional Response. *Nonlinear Analysis: Hybrid System*s,*1*(1),59-67.

Martin, A., & Ruan, S. (2001). Predator-Prey Models with Delay and Prey Harvesting. *Journal of Mathematical Biology,43*(3),247-267. DOI:10.1007/s002850100095

Mathews, J. H., & Fink, K. D. (2004). *Numerical Methods Using MATLAB* (4th ed., Vol. 4). Upper Saddle River, New Jersey, USA: Pearson Prentice Hall. Retrieved from http://vig.prenhall.com/.

Moghadas, S. M., Alexander, M. E., & Corbett, B. D. (2004). A Non-Standard Numerical Scheme for a   Generalized Gause-Type Predator-Prey Model. *Physica D: Nonlinear Phenomena*, *188*(1-2),134-151.

Mullen, Terry. (2015). An Analysis of Numerical Methods on Traffic Flow Models. In BSU Honors Program Theses and Projects. Item 105. Available at: http://vc.bridgew.edu/honors_proj/105

Paul, S., Mondal, S. P., & Bhattacharya, P. (2016). Numerical Solution of Lotka Volterra Prey Predator Model by Using Runge-Kutta-Fehlberg Method and Laplace Adomian Decomposition Method. *Alexandria   Engineering Journal, 55*(1), 613-617.

Rini, Y., & Alyauma, H. (2018). Numerical Solution of Nth-Order Fuzzy Differential Equations by Third Order Runge Kutta Method Based on Combination of Arithmatics, Harmonics and Geometrics Means. *Applied Mathematics, 8*(2), 19-25.

Ruan, S. (2009). On Nonlinear Dynamics of Predator-Prey Models with Discrete Delay. *Mathematical Modelling of Natural Phenomena, 4*(2), 140-188.

Safaa, J., Norihan, M., Raid, K., Fudziah, I., & Norfifah, B. (2016). Analysis of Ecological Model with Holling Type IV Functional Response. *International Journal of Pure and Applied Mathematics, 106*(1),317-331. DOI:10.12732/ijpam.v106i1.26

Sinha, T. K., & Chanda, S. (2014). Lotka Volterra Model with Time Delay. *International Journal of Mathematics Research, 6*(2), 115-120. Retrieved from http://www.irphouse.com

Xia, J., Liu, Z., Yuan, R., & Ruan, S. (2009). The Effects of Harvesting and Time Delay on Predator-Prey Systems with Holling Type II Functional Response. *SIAM Journal on Applied Mathematics, 70*(4),1178-1200.

Zill, D. G., & Cullen, M. R. (2009). *Differential Equations with Boundary-Value Problems* (7th ed.). United States: Brooks/Cole Cengage Learning.